A REPRESENTATION OF VECTOR AUTOREGRESSIVE PROCESSES WITH COMMON CYCLES

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ABSTRACT. We give necessary and sufficient conditions on the autoregressive polynomial for the existence of (possibly polynomial-) serial correlation common features as well as for other forms of common cycles. We characterize the resulting moving average representations. These conditions allow to define the restrictions on the VAR parameters that correspond to various form of common dynamics. Results are stated for stationary VAR processes and we indicate how they directly extend to cointegrated VAR systems integrated of order 1 and 2.

1. INTRODUCTION

Several macroeconomic theories predict the presence of common dynamic components in economic time series. This idea has been translated by Vahid and Engle (1993) and Engle and Kozicki (1993) into the notion of common features. Special features of economic time series are trends and cycles; some applications are contained in Kugler and Neusser (1993), Vahid and Issler (2002), Hecq, Palm, and Urbain (2002, 2006), Schleicher (2007) and Cubadda, Hecq, and Palm (2007), inter alia.

When trends are described by processes integrated of order 1, I(1), the notion of common trends corresponds to the property of co-integration. The relation between the Autoregressive (AR) and Moving Average (MA) representations for these processes is the object of Granger's representation theorem. Often vector autoregressive processes are found to

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describe well the dynamic properties of observed economic time series; Johansen proved a version of Granger's representation theorem for VAR processes, see Johansen (1996).

For the I(1) case, Johansen's version of Granger's representation theorem shows that co-integration implies a reduced-rank condition on the sum of VAR matrices and a full-rank condition of a different function of the same matrices. This corresponds to a complementary reduced-rank structure of the MA impact matrix in the MA representation of the first differences of the process.

Cyclical features have been associated with the presence of serially-correlated, stationary processes. Several definitions of common cycles have been proposed in the literature; some of them are related to the AR representation, such as the (polynomial) serial correlation common features of Vahid and Engle (1997) and Cubadda and Hecq (2001). Other definitions concern the MA representation; this is the case for instance of the notion of co-dependence introduced in Gourieroux and Peaucelle (1988), see also Vahid and Engle (1997). Recently Cubadda, Hecq, and Palm (2007) investigated the presence of common dynamics also in the final equations representation of the system.

For common cycles, there does not appear to be any equivalent to (Johansen's version of) Granger's representation theorem. Some questions that such a representation theory could answer include:

(1) If a VAR process presents serial correlation common features, what type of restrictions does this impose on its MA representation?
(2) If a VAR process presents co-dependence, what type of restrictions does this impose on its AR representation?
(3) Can the same VAR process display both serial correlation common features and co-dependence?
(4) If a VAR process presents serial correlation common features, what does this imply for its final equations?

This paper aims at providing such a representation theory for VAR processes, and in doing so, it answers the above questions. We restrict attention to VAR processes, as in Johansen (1996), because the VAR class appears to describe macroeconomic data well. Results are first derived for stationary processes, because cycles are associated with stationary autocorrelated processes. We then extend them to VAR processes integrated of order 1 and 2, showing how common trends and cycles can coexist, as expected from applications.

The conditions for serial correlation common features and for co-dependence are, similarly to the co-integration case, reduced rank conditions of some different functions of the VAR coefficients, with a terminal full-rank condition. The number of these reduced ranks is associated with an index m, which is a function of the degrees of the VAR, of its characteristic polynomial and of the adjoint polynomial.

We derive both necessary and necessary and sufficient conditions in terms of m and of the reduced rank conditions. We show that the same VAR can present both serial correlation common features and co-dependence. We also state conditions for commonality of the MA part in the final equation form. The present results provide tools of further analysis on the final equations with respect to Cubadda, Hecq, and Palm (2007), even though we do not consider the associated inference problem.

The rest of the paper is organized as follows: Section 2 introduces notation and definitions of structures of interest, while Section 3 collects the main representation results. Section 4 presents two worked-out examples, while Section 5 reports extensions and conclusions.

A final word on notation. In the following, $a := b$ and $b =: a$ indicate that a is defined by b; for any matrix polynomial $\pi(z) := \sum_{i=0}^{d_\pi} \pi_i z^i$ of dimension $p \times r$, where $r \leq p$, we indicate by $d_\pi := \deg \pi(z)$ its degree. For any full column rank matrix a, $\mathrm{col}(a)$ is the linear span of the columns of a, $a_\perp$ indicates a basis of $\mathrm{col}^\perp(a)$, the orthogonal complement of $\mathrm{col}(a)$. For any full column rank matrix a, $\bar{a}$ indicates $a(a'a)^{-1}$ and $P_a = \bar{a}a' = a\bar{a}'$ the orthogonal projector matrix onto $\mathrm{col}(a)$. For any reduced rank matrix A we often employ a rank-decomposition of the type $A = -\alpha\beta'$ where α and β are bases of $\mathrm{col}(A)$ and $\mathrm{col}(A')$, and the negative sign is chosen for convenience in the calculations. As it is well known, the choice of α and β in the rank decomposition is not unique, but all the results in the paper do not depend on the choice of basis.

2. Setup and definitions

We consider the vector autoregressive process of order d_Π

$$\Pi(L)X_t = \epsilon_t \tag{2.1}$$

where X_t is $p \times 1$, $\Pi(z) = \sum_{i=0}^{d_\Pi} \Pi_i z^i$ with $\Pi_0 = I_p$, is a $p \times p$ matrix polynomial and ϵ_t is a martingale difference sequence (with respect to the natural filtration generated by X_{t-s}, $s \geq 0$) and with positive definite conditional covariance matrix Ω. We are interested in the case of finite order VAR, $d_\Pi < \infty$.

Deterministic components are omitted from (2.1) for ease of exposition; they could be included by replacing X_t with $X_t - D_t$ or by replacing ϵ_t with $\epsilon_t + D_t$ in (2.1), where D_t are deterministic components. Here we exclude them in order to concentrate attention on conditions on $\Pi(z)$ that guarantee common dynamics.

We assume that the roots of $k(z) := \det \Pi(z)$ are outside the unit circle, which ensure the existence of a moving average (MA) representation. Thus the initial conditions $X_0, \ldots, X_{-d_\Pi+1}$ can be chosen so that solution of (2.1) is the linear process

$$X_t = C(L)\epsilon_t \tag{2.2}$$

where $C(z) := \operatorname{inv}\Pi(z) = \sum_{i=0}^{d_C} C_i z^i$ with $C(0) = I_p$. Here $d_C = \infty$ whenever $d_k \neq 0$, i.e. $k(z) := \det \Pi(z)$ is not constant.

The VAR system has also a final equations form, see e.g. Zellner and Palm (1974), Cubadda, Hecq, and Palm (2007):

$$k(L)X_t = K(L)\epsilon_t \tag{2.3}$$

where $K(z) := \operatorname{adj}\Pi(z)$ of degree d_K.

In order to define the common structures of interest we employ $p \times s$ matrix polynomials $\gamma(z)$ with $s \leq p$, $\gamma(z) := \sum_{i=0}^{d_\gamma} \gamma_i z^i$, where both γ_0 and γ_{d_γ} are assumed of full column rank. A matrix polynomial with this property is called of full column rank. We also assume that $\gamma(z)$ has finite order, $d_\gamma < \infty$.

The reason to consider full column rank matrix polynomials is that if γ_0 is $p \times s$ and not of full column rank, then $\gamma(z)$ can be substituted with $z^a \widetilde{\gamma}(z)$ where $a > 0$ and $\widetilde{\gamma}_0$ is of full column rank $s_1 < s$. Similarly if γ_{d_γ} is not full rank, $\gamma(z)$ can be substituted with $\widetilde{\gamma}(z)$ where $d_{\widetilde{\gamma}} < d_\gamma$ and $\widetilde{\gamma}_{d_{\widetilde{\gamma}}}$ is of full column rank $s_1 < s$. This leads to the following definitions.

Definition 2.1 (CS, CD and CE structures). *Let $\gamma(z) := \sum_{i=0}^{d_\gamma} \gamma_i z^i$ indicate a full column rank $p \times s$ matrix polynomial; if X_t in* (2.1) *satisfies*

$$\gamma'(L)X_t = \gamma_0'\epsilon_t$$

with $d_\gamma < d_\Pi$, then we say that X_t displays common serial correlation common features of order d_γ and indicate it with $X_t \in \mathrm{CS}_s(d_\gamma)$.
If X_t in (2.2) *satisfies*

$$\gamma_0' X_t = \gamma'(L)\epsilon_t$$

with $d_\gamma < d_C$, then we say that X_t displays co-dependence of order d_ϕ and indicate it with $X_t \in \mathrm{CD}_s(d_\gamma)$.
If X_t in (2.3) *satisfies*

$$k(L)\gamma_0' X_t = \gamma'(L)\epsilon_t$$

with $d_\gamma < d_K$, then we say that X_t displays commonality in the final equations of order (d_γ), and indicate it with $X_t \in \mathrm{CE}_s(d_\gamma)$.

Note how this definition encompasses several special cases, where here and in the following we omit the subscript s from CS_s unless needed for clarity. Serial correlation common features as introduced in Engle and Kozicki (1993) correspond to the case $\mathrm{CS}(0) = \mathrm{CD}(0)$. $\mathrm{CD}(d_\gamma)$ was introduced in Gourieroux and Peaucelle (1988), who considered only finite-order moving averages $d_C < \infty$. CD(1) structures were studied in Vahid and Engle (1997), see also the scalar component models in Tiao and Tsay (1989). Special cases of CS are given by the following notions: Polynomial Serial Correlation Common Features (PSCCF) which is defined in Cubadda and Hecq (2001) and discussed in Cubadda, Hecq, and Palm

(2007); Weak Form (WF) reduced rank structures, see Hecq, Palm, and Urbain (2006); Unpredictable Combinations (UP) defined in Paruolo (2006). Finally CE structures are considered in Cubadda, Hecq, and Palm (2007).

In the rest of the paper we show how the above different structures of interest are associated to restrictions on the coefficient matrices in $\Pi(z)$ of the VAR process.

3. Characterization of common cycles

In this section we present the main results of the paper. All proofs are placed in the Appendix. We first state conditions in Theorem 3.1 that link the degrees of the polynomials $\Pi(z)$, $k(z) := \det \Pi(z)$, $K(z) := \operatorname{adj} \Pi(z)$ with existence of CS, CD and CE. The result is stated in terms of the integer m defined as $m := d_\Pi + d_K - d_k$. These results give necessary ("order") conditions for CS, CD and CE in terms of m and d_Π.

Theorem 3.1 (Order conditions for CS, CD and CE). *Assume* $d_\Pi < \infty$ *and define* $m := d_\Pi + d_K - d_k$*; then*

- *i) the following conditions are equivalent:*
  - *i.1)* $m > 0$*;*
  - *i.2)* $X_t \in \mathrm{CS}(d_\gamma)$*, where* $d_\gamma \geq d_\Pi - m$*;*
  - *i.3)* $X_t \in \mathrm{CE}(d_\phi)$*, where* $d_\phi \geq d_k - d_\Pi$*.*
- *ii)* $X_t \in \mathrm{CD}(d_\phi)$ *if and only if* $m \geq d_\Pi$ *and in this case* $d_\phi \leq m - d_\Pi$*;*
- *iii) ii) implies i) but not viceversa.*

Theorem 3.1 shows that a CS or CE structure exists if and only if $m > 0$, i.e. when the last coefficient matrix of $\Pi(z)$ is singular. Moreover, if the degree of the determinant is greater than the degree of the adjoint of $\Pi(z)$, i.e. when $m < d_\Pi$, one only finds $\mathrm{CS}(d_\gamma)$ structures while in the opposite case, both $\mathrm{CS}(d_\gamma)$ and $\mathrm{CD}(d_\phi)$ structures coexist. In addition, note that $d_k - d_K = d_\Pi - m$ provides a lower bound for d_γ in $\mathrm{CS}(d_\gamma)$ and it reveals the highest reduction that can be achieved in the AR part. When this difference is negative, i.e. when $X_t \in \mathrm{CD}(d_\phi)$, it is trivially satisfied and does not provide any relevant information regarding the AR part. However, $d_K - d_k = m - d_\Pi$ provides an upper bound for d_ϕ in $\mathrm{CD}(d_\phi)$ and it reveals the lowest order that can be achieved in the MA part.

Hence the difference between the degrees of the adjoint and of the determinant of $\Pi(z)$ plays and important role to distinguish cases with $\mathrm{CS}(d_\gamma)$ structures from the cases with both $\mathrm{CS}(d_\gamma)$ and $\mathrm{CD}(d_\phi)$.

We next present a characterization of the restrictions that correspond to any given value of $m > 0$. These constraints are of reduced-rank type, and are stated in term of matrices α_i, β_i and Γ_i that are functions of the AR coefficients Π_i. This result is based on the recursive algorithm developed in Franchi (2007), see the proof in the Appendix.

Theorem 3.2 (Rank conditions)**.** *Assume $m > 0$ and let $i = 0, \cdots, m$ and $\Phi_i := M_i^\alpha \Gamma_i M_i^\beta$ where M_i^ϕ and Γ_i are defined below. Then*

$$\Phi_i = -\alpha_i \beta_i' \tag{3.1}$$

where α_i, β_i are $p \times r_i$ matrices of full column rank r_i, with $r_i < p - \sum_{h=0}^{i-1} r_h$ for $i < m$ and $r_m = p - \sum_{h=0}^{m-1} r_h$.
Conversely, if (3.1) is satisfied for $i = 0, \cdots, m^*$ with $r_{m^*} = p - \sum_{h=0}^{m^*} r_h$, then $m = m^*$.
The M_i^ϕ matrices are defined for $\phi := \alpha, \beta$ as $M_i^\phi := I - \sum_{h=0}^{i-1} P_{\phi_h}$, where $\sum_{i=a}^{b} := 0$ if $b < a$. The Γ_i are defined as $\Gamma_0 := \Pi_{d_\Pi}$, $\Gamma_{1,j} := \Pi_{d_\Pi - j}$ and $\Gamma_i := \Gamma_{i,1}^B$ where $\Gamma_{i,j}^B := Y_{i,j}$ for $2 \le i \le m+1$ is defined by the recursion

$$Y_{i,j} := Y_{i-1,j+1} + \sum_{h=0}^{i-2} F_{h,i,j} \tag{3.2}$$

where $F_{h,i,j} := \Gamma_{i-1} \bar{\beta}_h \bar{\alpha}_h' Y_{h+1,j}$.

The following remarks are in order

- The matrices $(\alpha_0 : \alpha_1 : \cdots : \alpha_{m-1} : \alpha_m)$ and $(\beta_0 : \beta_1 : \cdots : \beta_{m-1} : \beta_m)$ are square non-singular matrices with orthogonal blocks. To simplify notation, we split these matrices into blocks $b_i := (\beta_0 : \cdots : \beta_{i-1})$, $b_i^\circ := (\beta_i : \cdots : \beta_m)$, and similarly $a_i := (\alpha_0 : \cdots : \alpha_{i-1})$, $a_i^\circ := (\alpha_i : \cdots : \alpha_m)$.
- The conditions (3.1) are reduced-rank conditions for $i = 0, \cdots, m-1$, while the terminal condition for $i = m$ is a full-rank condition. The matrices $M_i^\alpha = P_{a_i^\circ}$, $M_i^\beta = P_{b_i^\circ}$ are orthogonal projectors, and the reduced-rank conditions (3.1) are "peeling away" subspaces until the terminal condition of full rank is met.

We are now in the position to state necessary and sufficient conditions on $\Pi(z)$ for the existence of CS in terms of the matrices α_i, β_i and Γ_i, $i = 0, \ldots, m$.

Theorem 3.3 (CS structures)**.** *Let φ denote a $p \times s$ full column rank matrix, and let $n \le d_\Pi - 1$. Then the following conditions are equivalent:*

a) *there exists $\mathrm{CS}_s(d_\Pi - n - 1)$;*
b) *$\varphi'(\Pi_{d_\Pi} : \cdots : \Pi_{d_\Pi - n}) = 0$ and $\varphi' \Pi_{d_\Pi - n - 1}$ of full row rank;*
c) *the following conditions are satisfied:*
   c.1) *(3.1) holds with $m \ge n + 1$,*
   c.2) *$\varphi'(\Gamma_1 b_1 : \cdots : \Gamma_n b_n) = 0$ and $\varphi \in \mathrm{col}(a_{n+1}^\circ)$,*
   c.3) *the conditions c.2) do not hold replacing n with $n + 1$.*

The next theorem provides necessary and sufficient conditions on $K(z) := \operatorname{adj} \Pi(z)$ for the existence of CE in terms of the matrices α_i, β_i and K_i, $i = 0, \ldots, m$, where the relation between K_i and α_i, β_i is given in (5.12) in the Appendix.

Theorem 3.4 (CE structures)**.** *Let φ denote a $p \times s$ full column rank matrix, and let $n \leq d_K - 1$. Then the following conditions are equivalent:*

a) *there exists* $\mathrm{CE}_s(d_K - n - 1)$*;*
b) $\varphi' (K_{d_K} : \cdots : K_{d_K - n}) = 0$ *and* $\varphi' K_{d_K - n - 1}$ *of full row rank;*
c) *the following conditions are satisfied:*
   c.1) (3.1) *holds with* $m \geq n + 1$
   c.2) $\varphi'(K_{d_K - 1} a_m^\circ : \cdots : K_{d_K - n} a_{m+1-n}^\circ) = 0$ *and* $\varphi \in \mathrm{col}(b_{m-n})$*,*
   c.3) *the conditions c.2) do not hold replacing n with $n + 1$.*

We finally turn to CD conditions. We note that Theorem 3.4 provides information on the CD structure because, from $C(z) = K(z)/k(z)$, one has

$$C_i = \sum_{h=0}^{\min(i, d_K)} K_h c_{i-h} \tag{3.3}$$

where $c_i = \frac{1}{i!} \left(\frac{d^i}{dz^i} \frac{1}{k(z)} \right) \Big|_{z=0}$ is a scalar. Hence the column and row structures of the K_h coefficients determines the ones in C_i; this allows to characterize the CD structures. A more thorough analysis of these implications will be included in a future revision of the paper.

4. Examples

Example 1 ($\mathrm{CS}(d_\gamma)$ *and* $\mathrm{CD}(d_\phi)$ *structures).* Consider the two stationary VAR(2) processes

$$\Pi_j(L) X_t^j = \epsilon_t, \; j = A, B,$$

where X_t^j and $\epsilon_t \sim i.i.d.(0, \Omega)$ have dimensions 2×1,

$$\Pi_A(z) := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1/2 \end{pmatrix} z + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} z^2$$

and

$$\Pi_B(z) := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1/2 \end{pmatrix} z + \begin{pmatrix} 1/2 & 1 \\ 0 & 0 \end{pmatrix} z^2,$$

share the same AR equation for the second variable and the left null space of Π_2^A and Π_2^B are identical. Hence, defining $\gamma_0 = (0 : 1)'$ one has

$$\gamma_0' \Pi_A(z) = \gamma_0' \Pi_B(z) = (0 : 1) + (0 : 1/2) z.$$

The MA representations

$$X_t^j = \sum_{n=0}^{\infty} C_n^j \epsilon_{t-n}, \; j = A, B,$$

have matrix coefficients

$$C_0^A = C_0^B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } C_1^A = C_1^B = \begin{pmatrix} 0 & 0 \\ 0 & -1/2 \end{pmatrix}$$

and for $i = 2, 3, \ldots$

$$C_i^A = c_i \begin{pmatrix} 0 & -1 \\ 0 & 1/4 \end{pmatrix} \text{ where } c_i \text{ is a scalar and } C_i^B \text{ is } \begin{cases} \text{non singular if } i \text{ is even} \\ \text{singular if } i \text{ is odd.} \end{cases}$$

Then $\phi_0 = (1 : 4)'$ is such that

$$\phi_0' X_t^A = \phi_0' \epsilon_t + \phi_1' \epsilon_{t-1}$$

but

$$\phi_0' X_t^B = \sum_{i=0}^{\infty} \phi_i' \epsilon_{t-i}$$

for any ϕ_0. Hence $X_t^A \in \mathrm{CD}(1)$ and $X_t^B \in \mathrm{CS}(1)$.

This could be understood right away from the VAR coefficients by checking the degrees of the determinant and the adjoint of

$$\Pi_A(z) = \begin{pmatrix} 1 & z^2 \\ 0 & 1 + z/2 \end{pmatrix} \text{ and } \Pi_B(z) = \begin{pmatrix} 1 + z^2/2 & z^2 \\ 0 & 1 + z/2 \end{pmatrix};$$

in fact one has

$$\det \Pi_A(z) = 1 + z/2 \text{ and } \operatorname{adj} \Pi_A(z) = \begin{pmatrix} 1 + z/2 & -z^2 \\ 0 & 1 \end{pmatrix}$$

so that $d_{det_A} := \deg \det \Pi_A(z) = 1$, $d_{adj_A} := \deg \operatorname{adj} \Pi_A(z) = 2$, $m_A - d_{\Pi_A} = d_{adj_A} - d_{det_A} = 1$ and then $X_t^A \in \mathrm{CD}(d_\phi)$ where $d_\phi \leq 1$. Similarly for process X_t^B one finds $d_{det_B} = 3$, $d_{adj_B} = 2$ and $d_{\Pi_B} - m_B = d_{det_B} - d_{adj_B} = 1$ so that $X_t^B \in \mathrm{CS}(1)$ because $d_\gamma = d_{\det \Pi} - d_{\mathrm{adj}\, \Pi} = d_\Pi - 1 = 1$.

Example 2 (Necessity of the reduced rank conditions). The stationary VAR(2) process $\Pi(L) X_t = \epsilon_t$ where X_t and $\epsilon_t \sim i.i.d.(0, \Omega)$ have dimension 3×1 and

$$\Pi(z) := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} * & 0.5 & 0.8 \\ * & 0.4 & 0.7 \\ * & 0.3 & 0.6 \end{pmatrix} z + \begin{pmatrix} 0.5 & 0 & 0 \\ 0.5 & 0 & 0 \\ 0.5 & 0 & 0 \end{pmatrix} z^2,$$

where $*$ is understood to be 0, is such that $d_{\det \Pi} = d_{\mathrm{adj}\, \Pi} = 3$, so that $m = d_\Pi = 2$ and $X_t \in \mathrm{CD}(d_\gamma)$ where $d_\gamma \geq 0$. From Π_2, see Theorem 3.2, one finds $\alpha_0 = (1 : 1 : 1)'$ and $\beta_0 = c_0 (1 : 0 : 0)'$ where c_0 is a scalar so that $\alpha_0' \Pi_2 \neq 0$ and $\deg \alpha_0' \Pi(z) = 2$. From $\Phi_1 = P_{\alpha_{0\perp}} \Pi_1 P_{\beta_{0\perp}}$ one has $\alpha_1 = (3 : -1 : -2)'$ and $\beta_1 = c_1 (0 : 1 : 1)'$ where c_1 is a scalar; because $\alpha_1' \Pi_2 = 0$ and $\alpha_1' \Pi_1 \neq 0$ one has $\deg \alpha_1' \Pi(z) = 1$. Finally, $\Phi_2 = (I - P_{\alpha_0} - P_{\alpha_1})(I - P_{\beta_0} - P_{\beta_1})$ gives $\alpha_2 = (1 : -2 : -1)'$, such that $\mathrm{col}(\alpha_0 : \alpha_1 : \alpha_2) = I$ and $\alpha_2' \Pi_2 = \alpha_2' \Pi_1 = 0$ so that one has $\deg \alpha_2' \Pi(z) = 0$. Note that if a $*$ coefficient in Π_1 is different from 0, even though one has the same m and the same α_i and β_i coefficients, condition c.2) in Theorem 3.2 fails and thus one finds $\deg \alpha_2' \Pi(z) = 1$.

5. Extensions and conclusion

The present results allow to characterize restrictions on the VAR coefficients that correspond 1-to-1 to the presence of common dynamic features of the CF, CD, CFE type. It is shown that all these characteristics are associated with the existence of reduced rank structures of some functions of the VAR coefficient matrices.

The present results are formulated in terms of stationary VAR processes, but they can be extended e.g. to the case of VARs with integrated series of order 1, I(1). In fact, when some $p \times 1$ process Y_t is I(1), then Johansen's version of Granger's representation theorem, see Johansen (1996), can be used to show that $X_t := (Y_t' \beta_Y : \Delta Y_t' \beta_{Y\perp})'$ is a stable VAR system, where $\Delta := 1 - L$ is the difference operator and β_Y is the cointegrating matrix.

Hence all results obtained here extend directly to I(1) VARs. A similar result is obtained in Johansen's representation theorem for I(2) VAR processes, see Johansen (1996). Similar results hold for systems integrated of higher order, see Franchi (2007). We can hence conclude that the present results stated for stable VARs can be used to characterize common dynamics for I(d) VAR processes, for generic $d = 1, 2, \ldots$.

Appendix

In order to study the behavior of $\Pi(z)$ at ∞, we map ∞ into 0 by introducing the transformation $z \mapsto 1/z$ and define the following † and ‡ operators[1]:

$$\Pi_\dagger(z) := \Pi(z^{-1}) \qquad \text{and} \qquad \Pi_\ddagger(z) := z^{d_\Pi}\Pi_\dagger(z) = z^{d_\Pi}\Pi(z^{-1}). \tag{5.1}$$

Note that $\Pi_\ddagger(z) = \Pi_{d_\Pi} + \Pi_{d_\Pi - 1}z + \cdots + \Pi_0 z^{d_\Pi}$ contains the same coefficients of $\Pi(z)$ in reversed order. For ease of notation we indicate $A(z) := \Pi_\ddagger(z)$.

5.1. **Order conditions.** Recall that $k(z) := \det \Pi(z)$ and $K(z) := \operatorname{adj} \Pi(z)$. The following proposition relates m to the order of the pole of $\operatorname{inv} \Pi_\ddagger(z)$ at $z = 0$.

Proposition 5.1. *Let $\Pi_{d_\Pi} \neq 0$ be singular; then*

$$\operatorname{inv} A(z) = z^{-m}\frac{K_\ddagger(z)}{k_\ddagger(z)} \tag{5.2}$$

where $K_\ddagger(0)$, $k_\ddagger(0) \neq 0$ and $m := d_\Pi + d_K - d_k > 0$.

Proof. Note that $A(0) = \Pi_{d_\Pi}$ is singular, i.e. $\det A(0) = 0$, which shows that 0 is a root of $\det A(z)$, so that $\det A(z) = z^a g(z)$ where $a > 0$ and $g(0) \neq 0$. Let z_i be such that $A(1/z_i)$ is singular, i.e. let $g(z) = (z - 1/z_i)^{m_i} l(z)$ where $m_i > 0$ and $l(1/z_i) \neq 0$. Because $A(1/z_i)$ is mapped into $\Pi(z_i)$ by the ‡ operator, it follows that $\Pi(z_i)$ is singular and hence $k(z) = (z - z_i)^{m_i} h(z)$ where $h(z_i) \neq 0$. Because $\Pi(0) = I$, one has $k(0) \neq 0$ and this implies that the roots of $g(z)$ are all finite and they are the reciprocals of the roots of $k(z)$. Hence one has

$$\det A(z) = z^a k_\ddagger(z), \; k_\ddagger(0) \neq 0, \tag{5.3}$$

where $a > 0$ and, similarly,

$$\operatorname{adj} A(z) = z^b K_\ddagger(z), \; K_\ddagger(0) \neq 0, \tag{5.4}$$

where $0 \leq b < a$, because if the rank of $A(0) = \Pi_{d_\Pi}$ is less than $p - 1$ all co-factors in $\operatorname{adj} A(0)$ are 0 and one can factor z^b from the adjoint; in case the rank of Π_{d_Π} is $p - 1$ there is no such factor. This implies $\operatorname{inv} A(z) = z^{-c}\frac{K_\ddagger(z)}{k_\ddagger(z)}$ where $K_\ddagger(0)$, $k_\ddagger(0) \neq 0$, i.e. $\operatorname{inv} A(z)$ has a pole of order c at 0, $c := a - b > 0$.

We next wish to show that $c = m$. By the definition of the † and ‡ operators one has $\operatorname{inv} \Pi_\dagger(z) = z^{d_\Pi} \operatorname{inv} \Pi_\ddagger(z)$; thus, by the above derivations one finds

$$\operatorname{inv} \Pi_\dagger(z) = z^{d_\Pi - c}\frac{K_\ddagger(z)}{k_\ddagger(z)}, \tag{5.5}$$

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[1]Stability conditions of autoregressive processes are stated equivalently in terms either of the roots of $\det \Pi(z)$ or in terms of the roots of $\det \Pi_\ddagger(z)$, see e.g. Priestley (1981) pages 133 and 686.

and because $\operatorname{inv}\Pi(z) = \frac{K(z)}{k(z)}$, one also has

$$\operatorname{inv}\Pi_\dagger(z) = \frac{K_\dagger(z)}{k_\dagger(z)} = z^{d_k - d_K}\frac{K_\ddagger(z)}{k_\ddagger(z)}. \tag{5.6}$$

Equating (5.5) and (5.6) one finds $c - d_\Pi + d_K - d_k = 0$, i.e. $c = m$. □

5.2. **Rank conditions.** For ease of notation we indicate $A(z) := \Pi_\ddagger(z)$, $G(z) = K_\ddagger(z)$ and $g(z) = k_\ddagger(z)$. The algorithm described below builds on the identity $A(z)\operatorname{adj}A(z) = \operatorname{adj}A(z)A(z) = \det A(z)I_p$ which gives

$$B(z) = H(z) = z^m g(z)I_p \tag{5.7}$$

where $B(z) := A(z)G(z)$, $H(z) := G(z)A(z)$ and $d_B = d_H = m + d_g$. In (5.7) one has $B_j = H_j$, and we use the notation $B_j := \sum_{h=0}^{j} A_h G_{j-h}$ and $H_j := \sum_{h=0}^{j} G_{j-h}A_h$; then one has

$$B_j = H_j = g_{j-m} I\, 1(j \geq m) \qquad \text{for } j = 0, \cdots, m + d_g \tag{5.8}$$

where $1(\cdot)$ is the indicator function, $g_0 \neq 0$ by (5.3) and $g_{d_g} \neq 0$ by definition of the determinant.

For the proof of Theorem 3.2 and Lemma 5.3 below we make repeated use of the following result.

Lemma 5.2. *Let $G \neq 0$ and assume $\Phi G = G\Phi = 0$; this implies*

i) $\Phi = -\alpha\beta'$ with α and β of dimension $p \times r$ and full column rank, $\operatorname{col}(G) \subseteq \operatorname{col}^\perp(\beta)$ *and* $\operatorname{col}(G') \subseteq \operatorname{col}^\perp(\alpha)$*;*

ii) If moreover $\operatorname{col}(G) \subseteq \operatorname{col}(b)$ *and* $\operatorname{col}(G') \subseteq \operatorname{col}(a)$*, then i) implies* $\operatorname{col}(G) \subseteq \mathcal{B} := \operatorname{col}(b) \cap \operatorname{col}^\perp(\beta)$ *and* $\operatorname{col}(G') \subseteq \mathcal{A} := \operatorname{col}(a) \cap \operatorname{col}^\perp(\alpha)$ *where, because $G \neq 0$,* $\dim\mathcal{A} = \dim\mathcal{B} > 0$.

Proof. i) Because $\det\Phi \neq 0$ implies $G = 0$, one has $\Phi = -\alpha\beta'$ with α and β of dimension $p \times r$ and full column rank $r < p$, so that $\operatorname{col}(G) \subseteq \operatorname{col}^\perp(\beta)$ and $\operatorname{col}(G') \subseteq \operatorname{col}^\perp(\alpha)$;

ii) From $\operatorname{col}(G) \subseteq \operatorname{col}(b)$ and *i)* one has $\operatorname{col}(G) \subseteq \mathcal{B} := \operatorname{col}(b) \cap \operatorname{col}^\perp(\beta)$; and because $G \neq 0$ it must be that $\mathcal{B} \neq \{0\}$ and thus $\dim\mathcal{B} > 0$. Similarly one shows that $\operatorname{col}(G') \subseteq \mathcal{A} := \operatorname{col}(a) \cap \operatorname{col}^\perp(\alpha)$ and $\dim\mathcal{A} > 0$. □

Proof of Theorem 3.1. Let $K(z) = \operatorname{adj}\Pi(z)$ and $k(z) = \det\Pi(z)$; then

i.1) ⇒ i.2) and i.3). If $m > 0$ then (5.8) and Lemma 5.2 imply $\det\Pi_{d_\Pi} = 0$ and $\det K_{d_K} = 0$. Then there exist $\gamma_0 \neq 0$ such that $\gamma_0'\Pi(z) = \gamma'(z)$ where $0 \leq d_\gamma \leq d_\Pi - 1$. Then $k(z)\gamma_0' = \gamma'(z)K(z)$ follows from the last equation and one has $d_k \leq d_\gamma + d_K$. Similarly, $\det K_{d_K} = 0$ implies $\phi_0'K(z) = \phi'(z)$ where $0 \leq d_\phi \leq d_K - 1$ and $\phi_0 \neq 0$. Then $k(z)\phi_0' = \phi'(z)\Pi(z)$ follows from the last equation and one has $d_k \leq d_\phi + d_\Pi$.

i.2) $\Rightarrow$ *i.1).* (By contradiction) Assume $m = 0$ and Π_{d_Π} of reduced rank; because (5.8) for $j = m = 0$ gives $\Pi_{d_\Pi} K_{d_K} = g_0 I$ and implies $\det \Pi_{d_\Pi} \neq 0$ and $\det K_{d_K} \neq 0$, we reach a contradiction and thus it must be $m > 0$. Similarly one shows that *i.3)* $\Rightarrow$ *i.1).*

ii) Because $X_t \in \mathrm{CD}(d_\phi)$ if and only if $\phi_0' C(z) = \phi'(z)$, i.e $\phi_0' K(z) = k(z)\phi'(z)$, we need to show that $\phi_0' K(z) = k(z)\phi'(z)$ if and only if $d_K \geq d_k$. *Suff.* If $\phi_0' K(z) = k(z)\phi'(z)$ then $d_K \geq d_k + d_\phi$ implies $d_K \geq d_k$ because $d_\phi \geq 0$. *Nec.* (By contradiction) Assume $d_k > d_K$ and $\phi_0' K(z) = k(z)\phi'(z)$; because the degree of the r.h.s. is $d_k + d_\phi$ where $d_\phi \geq 0$ and the degree of the l.h.s. is at most d_K we reach a contradiction and thus it must be $d_K \geq d_k$.

iii) Direct consequence of *i)* and *ii)*. This completes the proof. □

Lemma 5.3. *In the notation of Theorem* 3.2*, consider indices* i *and* s *with* $0 \leq i \leq m$ *and* $0 \leq s \leq m + d_g$*; the following equalities hold*

$$-\alpha_i \beta_i' G_s + M_i^\alpha \sum_{h=1}^{s} \Gamma_{i+1,h}^B G_{s-h} + M_i^\alpha \Theta_{i+1,s}^B = 0 \tag{5.9}$$

$$-G_s \alpha_i \beta_i' + \sum_{h=1}^{s} G_{s-h} \Gamma_{i+1,h}^H M_i^\beta + \Theta_{i+1,s}^H M_i^\beta = 0 \tag{5.10}$$

where $\sum_{i=a}^{b} \cdot := 0$ *if* $b < a$*. Here* $\Gamma_{1,h}^H := \Gamma_{1,h}^B$ *and, for* $i \geq 2$*,* $\Gamma_{i,j}^H := Y_{i,j}$ *in* (3.2) *for* $F_{h,i,j} := \Gamma_{h+1,j}^H \bar{\beta}_h \bar{\alpha}_h' \Gamma_{i-1,1}^H$*. Define also* $\Theta_{1,s}^B := -g_{s-m} 1(s \geq m) I =: \Theta_{1,s}^H$ *and for* $i \geq 2$*,* $\Theta_{i,j}^B := Y_{i,j}$ *in* (3.2) *for* $F_{h,i,j} := \Gamma_{i-1} \bar{\beta}_h \bar{\alpha}_h' Y_{h+1,j}$*. Finally, for* $i \geq 2$*,* $\Theta_{i,j}^H := Y_{i,j}$ *in* (3.2) *for* $F_{h,i,j} := Y_{h+1,j} \bar{\beta}_h \bar{\alpha}_h' \Gamma_{i-1}$*.*

Moreover, $G_0 = -g_0 \bar{\beta}_m \bar{\alpha}_m'$ *and for* $i = 0, \ldots, m$

$$\bar{a}_i^{\circ\prime} \Gamma_i \bar{b}_i^\circ = \begin{pmatrix} -g_0 I_{r_i} & 0 \\ 0 & 0 \end{pmatrix} \tag{5.11}$$

$$\bar{b}_{m-i+1}' G_i \bar{a}_{m-i+1} = \begin{pmatrix} 0 & 0 \\ 0 & -g_0 I_{r_{m-i}} \end{pmatrix}. \tag{5.12}$$

Proof of Theorem 3.2 and Lemma 5.3. We want to show that for $i < m$ one has $\Phi_i G_0 = G_0 \Phi_i = 0$ and thus one can apply Lemma 5.2 to obtain (3.1). We also want to prove (5.9) and (5.10) recursively for $i = 0, \ldots, m$. Finally we want to show that $\Phi_m G_0 = g_0 P_{\alpha_m}$ and $G_0 \Phi_m = g_0 P_{\beta_m}$, which imply (3.1) for $i = m$ and $G_0 = -g_0 \bar{\beta}_m \bar{\alpha}_m'$.

For $i = 0$, $B_0 = H_0 = 0$ and one has $\Phi_0' G_0 = G_0 \Phi_0 = 0$ so that Lemma 5.2 applies because $G_0 \neq 0$, see (5.4). Hence (3.1) holds for $i = 0$ and one has $G_0 = M_1^\beta G_0 M_1^\alpha$ and $r_0 < p$, see *ii).* Moreover, (5.9) and (5.10) for $i = 0$ follow from substituting $A_0 = \Phi_0 = -\alpha_0 \beta_0'$ and the definition of $\Gamma_{1,j}^B$, $\Theta_{1,j}^B$, $\Gamma_{1,j}^H$, and $\Theta_{1,j}^H$ into the definition of B_j and H_j respectively, and using (5.8).

Next we show that if (3.1), (5.9) and (5.10) hold for $i = 0, \dots, n$, then the same equations hold also for $i = n + 1$, where n is chosen $0 \le n \le m - 1$. Take $s = 0$ and $i = n < m - 1$ in (5.9) and (5.10) and note that $\Theta^B_{n+1,0} = \Theta^H_{n+1,0} = 0$. Pre-multiplying (5.9) by M^α_{n+1} and post-multiplying (5.10) by M^β_{n+1}, one finds

$$M^\alpha_{n+1}\Gamma_{n+1}G_0 = G_0\Gamma_{n+1}M^\beta_{n+1} = 0. \tag{5.13}$$

Next observe that by the induction assumption we have $G_0 = M^\beta_{n+1}G_0M^\alpha_{n+1}$; substituting into (5.13) one finds $\Phi_{n+1}G_0 = G_0\Phi_{n+1} = 0$ so that Lemma 5.2 applies. Hence (3.1) holds for $i = n + 1$ with $r_{n+1} < p - \sum_{h=0}^{n} r_h$, see *ii)*, and $G_0 = M^\beta_{n+2}G_0M^\alpha_{n+2}$.

Next we prove the update of (5.9); consider first (5.9) for any $0 \le i \le n$ and pre-multiply it by $-\bar{\alpha}'_i$ to find

$$\beta'_i G_s = \bar{\alpha}'_i \sum_{h=1}^{s} \Gamma^B_{i+1,h} G_{s-h} + \bar{\alpha}'_i \Theta^B_{i+1,s}. \tag{5.14}$$

Next consider (5.9) for $s = n + 1$ and $i = n$; pre-multiply it by M^α_{n+1} to find

$$0 = M^\alpha_{n+1}\Gamma_{n+1}G_s + M^\alpha_{n+1}\sum_{h=1}^{s}\Gamma^B_{n+1,h+1}G_{s-h} + M^\alpha_{n+1}\Theta^B_{n+1,s+1} =: a + b + c \text{ (say)}. \tag{5.15}$$

Using the projection identity $I = M^\beta_{n+1} + \sum_{h=0}^{n} P_{\beta_h}$, substituting from (5.13) and (5.14), and rearranging terms one finds

$$a = -\alpha_{n+1}\beta'_{n+1}G_s + M^\alpha_{n+1}\sum_{l=1}^{s}(\Gamma_{n+1}\sum_{h=0}^{n}\bar{\beta}_h\bar{\alpha}'_h\Gamma^B_{h+1,l})G_{s-l} + M^\alpha_{n+1}\Gamma_{n+1}\sum_{h=0}^{n}\bar{\beta}_h\bar{\alpha}'_h\Theta^B_{h+1,s}.$$

Summing $a + b + c$ one finds (5.9) for $i = n + 1$. Similarly one obtains the update of (5.10). This proves (3.1), (5.9) and (5.10) for $i = 0, \dots, m - 1$.

Finally, consider $i = m-1$ and take $s = 1$ in (5.9) and (5.10). Pre-multiplying (5.9) by M^α_m and post-multiplying (5.10) by M^β_m, one finds $M^\alpha_m\Gamma_mG_0 = g_0M^\alpha_m$ and $G_0\Gamma_mM^\beta_m = g_0M^\beta_m$. Substituting $G_0 = M^\beta_mG_0M^\alpha_m$, derived from the induction step, for $n = m - 1$ one has

$$\Phi_mG_0 = g_0M^\alpha_m \qquad \text{and} \qquad G_0\Phi_m = g_0M^\beta_m. \tag{5.16}$$

From the first equation one finds $\text{col}(\Phi_m) = \text{col}(M^\alpha_m) = \text{col}(\alpha_m)$, where α_m is a basis of $\text{col}(M^\alpha_m)$ of dimension $r_m = p - \sum_{h=0}^{m-1} r_h$; hence $\Phi_m =: -\alpha_m\beta'_m$ is the corresponding rank decomposition and (3.1) holds for $i = m$. Substituting $\Phi_m = -\alpha_m\beta'_m$ in (5.16) one obtains (5.9) and (5.10) for $i = m$. Because $\beta'_mG_0\alpha_m = -g_0I_{r_m}$, one finds $G_0 = P_{\beta_m}G_0P_{\alpha_m} = -g_0\bar{\beta}_m\bar{\alpha}'_m$. Note that $M^\gamma_i = \sum_{h=i}^{m} P_{\gamma_h}$ and that, setting $j, n \ge i$ one has $\bar{\alpha}'_j\Gamma_i\bar{\beta}_n = \bar{\alpha}'_jM^\alpha_i\Gamma_iM^\beta_i\bar{\beta}_n = \bar{\alpha}'_j\Phi_i\bar{\beta}_n = -\bar{\alpha}'_j\alpha_i\beta'_i\bar{\beta}_n$ which proves (5.11).

In order to prove (5.12) we proceed by induction. Because $G_0 = -g_0\bar{\beta}_m\bar{\alpha}'_m$, (5.12) holds for $i = 0$. Next we assume (5.12) for $i = 0, \dots, n$ and show that it holds for $i = n + 1 \le m$.

From (5.9) with $j = i + n + 1$ and $i \leq m - n - 1$ one has

$$\beta_i' G_{n+1} = \bar{\alpha}_i' \sum_{h=1}^{n+1} \Gamma_{i+1,h}^B G_{n+1-h} - \delta_{i,m-n-1} g_0 \bar{\alpha}_i'$$

so that $\beta_i' G_{n+1} \alpha_j = 0$ for $j = 0, \ldots, m - n - 1$ holds by the induction assumption and $\beta_{m-n-1}' G_{n+1} \alpha_{m-n-1} = -g_0 I_{r_{m-n-1}}$ follows from the last equation. Similarly from (5.10) one has $\beta_{m-n-1}' G_{n+1} \alpha_j = 0$ for $j = 0, \ldots, m - n - 2$ and thus (5.12) holds for $i = n + 1$ and this completes the proof. □

Corollary 5.4. *Consider indices i and j with $1 \leq i \leq m + 1$ and $0 \leq j \leq m + d_g$; then $\Theta_{i,j} = 0$ for $i + j \leq m$ and $\Theta_{i,j} = -g_0 I$ for $i + j = m + 1$. Moreover,* $\mathrm{col}(G_s) = \mathrm{col}(b_{m-s}^{\circ})$ *and* $\mathrm{col}(G_s') = \mathrm{col}(a_{m-s}^{\circ})$, *i.e.* $G_s = \bar{b}_{m-s}^{\circ} H_s \bar{a}_{m-s}^{\circ\prime}$.

Proof. By definition $\Theta_{1,j} = 0$ for $0 \leq j \leq m - 1$ and $\Theta_{1,m} = -g_0 I$; then using (3.2) one has $\Theta_{2,j} = 0$ for $0 \leq j \leq m - 2$ and $\Theta_{2,m-1} = \Theta_{1,m} = -g_0 I$. By iterating (3.2) one finds the statement. PROVE THE SECOND STATEMENT! □

Proof of Theorem 3.3. Let $A(z) = \Pi_{\ddagger}(z)$; then

a) ⇔ b) by definition.

b) ⇒ c.1) Because $\varphi' A(z) = z^{n+1} \phi'(z)$ where $\phi(0)$ has full column rank, from $A(z)G(z) = z^m g(z) I$ one has $z^{n+1} \phi'(z) A(z) = z^m g(z) \varphi'$ so that $m \geq n + 1$.

b) ⇒ c.2) Because $A_0 = \Gamma_0$, $\varphi' A_0 = 0$ implies $\varphi' \Gamma_0 = 0$. Next we proceed by induction, assuming that $\varphi' A_i = 0$ implies $\varphi' \Gamma_i b_i = 0$ and $\varphi \in \mathrm{col}(a_{i+1}^{\circ})$ for $i = 0, \ldots, q$ and proving it for $i = q + 1$, where $1 \leq q \leq n - 1$. Consider (3.2) for $i = q + 1$ and note that $\varphi' F_{h,q+1,j} = \varphi' \Gamma_q \bar{\beta}_h \bar{\alpha}_h' \Gamma_{h+1,j}^B = 0$ for $0 \leq h \leq q - 1$ by the induction assumption. Hence $\varphi' \Gamma_{q+1} = \varphi' \Gamma_{1,q+1}^B = \varphi' A_{q+1} = 0$. This implies that for $i = 0, \ldots, n$ one has $\varphi' \Gamma_i b_i = 0$; moreover, because $\varphi \in \mathrm{col}(a_{q+1}^{\circ})$, $0 = \varphi' \Gamma_{q+1} M_{q+1}^{\beta} = \varphi' \Phi_{q+1} = -\varphi' \alpha_{q+1} \beta_{q+1}'$, i.e. $\varphi \in \mathrm{col}(a_{q+2}^{\circ})$.

b) ⇒ c.3) From (3.2) one has $\varphi' \Gamma_{n+1} = \varphi' A_{n+1}$ because $\varphi' F_{h,n+1,j} = 0$ as above so that $\varphi' A_{n+1}$ of full row rank implies $\varphi' \Gamma_{n+1}$ of full row rank; hence $\varphi' \Gamma_{n+1} b_{n+1}$ has full row rank.

c) ⇒ b) Assume $\varphi' \Gamma_i b_i = 0$, $\varphi \in \mathrm{col}(a_{n+1}^{\circ})$; then $\varphi' \Gamma_i b_i \bar{b}_i' = 0$, $\varphi' a_i \bar{a}_i' \Gamma_i M_i^{\beta} = 0$ and $\varphi' M_i^{\alpha} \Gamma_i M_i^{\beta} = \varphi' \Phi_i = 0$. Summing these 3 terms, one finds $\varphi' \Gamma_i = 0$. Pre-multiplying (3.2) by φ', one finds $\varphi' \Gamma_{i,1}^B = \varphi' \Gamma_{i-1,2}^B$ because $\varphi' F_{h,i,j} = \varphi' \Gamma_{i-1} \bar{\beta}_h \bar{\alpha}_h' \Gamma_{h+1,j}^B = 0$ for $0 \leq h \leq i - 2$ by assumption. Iterating one finds $\varphi' \Gamma_i = \varphi' \Gamma_{1,i}^B = \varphi' A_i$ and this proves $\varphi' (A_0 : \cdots : A_n) = 0$, noting that $\varphi' \Gamma_0 = \varphi' A_0$. If $\varphi' \Gamma_{n+1} b_{n+1}$ has full row rank, then $\varphi' \Gamma_{n+1} = \varphi' A_{n+1}$ from (3.2) has full row rank. If $\varphi \in \mathrm{col}(\alpha_{n+1})$, then $\varphi' \Gamma_{n+1}$ has full row rank, see (5.11), and this completes the proof. □

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