INVESTMENT EFFICIENCY IN THE PRESENCE OF RESTRICTIONS

ALESSANDRO BUCCIOL*
University of Padua

RAFFAELE MINIACI
University of Brescia

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Abstract

We consider as a measure of efficiency the wealth compensative variation necessary to make the investor indifferent between the observed investment and the optimal restricted portfolio. We derive the asymptotic distribution of this measure, its finite sample properties, and propose an efficiency test. We suggest a procedure to estimate the implicit risk aversion as the level for which the wealth loss is minimized. In our illustrative empirical application we consider two mean-variance inefficient strategies to show that for reasonable levels of RRA we cannot reject the hypothesis that these investments are consistent with an optimizing behavior.

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* Corresponding author: Alessandro Bucciol, University of Padua, Dept. of Economics "Marco Fanno", Via del Santo 33, 35123 Padova (ITALY). Telephone: +39 349.428.1570. Fax: +39 049.827.4211.
Email: alessandro.bucciol@unipd.it.
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1. Introduction

The efficiency of an investment is usually assessed by means of a standard mean-variance approach. In the simplest case of no restrictions on portfolio shares, such a framework implies that the performance of any investment is measured in terms of its Sharpe ratio, referred to as "the most common measure of risk-adjusted return" by Modigliani and Modigliani (1997). Using this measure, several statistical tests have been developed to establish the efficiency of an investment; among others, those proposed by Jobson and Korkie (henceforth JK, 1982), Gibbons et al. (1989), and Gourieroux and Jouneau (GJ, 1999) are noteworthy.

The use of the Sharpe ratio is relatively simple and intuitive but lacks some important features. Above all, it is not possible to take account of market imperfection when building the optimal portfolio weights. The widespread use of Sharpe ratios depends on the well-known fact that their upper limit is reached by any portfolio in the mean-variance efficient frontier built as a combination of the market portfolio and the risk free asset. Such a frontier is derived disregarding market imperfections, but in their presence it would assume a different shape.

In particular, two kinds of constraints are relevant: transaction costs and inequality constraints. Transaction costs are costs incurred when buying or selling assets. They include brokers' commissions and spreads, i.e., the difference between the price paid for an asset and the price it can be sold. Over a sufficiently short time horizon, transaction costs make the asset illiquid and are, therefore, equivalent to equality constraints on the portfolio weights from an investor's perspective. Although transaction costs may be negligible in the case of financial assets, they are instead relevant for real assets (Grossman and Laroque, 1990). An optimizing investor chooses, then, the composition of her financial portfolio conditional on the stock held in real assets. The most relevant case is that of housing (Flavin, 2002, and Pelizzon and Weber, 2003); following in particular GJ we know that, when equality constraints on some portfolio weights are taken into account, it is however possible to translate the original plane in another mean-variance frontier, conditional on the constrained assets.

Another important market imperfection is represented by inequality constraints. In actual stock markets, for instance, short-selling is not prohibited, but discouraged by the fact that the proceeds are not normally available to be invested elsewhere; this is enough to eliminate a private investor with just mildly negative beliefs (Figlewski, 1981). On the contrary, mutual fund constraints are widespread and may be seen as one component of the set of monitoring mechanisms that reduce the costs arising from frictions in the principal-agent relation (Almazan et al., 2004). Considering these constraints, we would be faced with a different frontier of feasible portfolios, of unknown shape, whose relationship with the

Sharpe ratio is not clear. With only short-sales restrictions in particular, there may be switching points along the mean-variance frontier corresponding to changes in the set of assets held. Each switching point corresponds to a kink (Dybvig, 1984), and the mean-variance frontier consists then of parts of the unrestricted mean-variance frontiers computed on subsets of the primitive assets.

Although the issue has been often discussed in the literature, empirical works usually make use of the standard, unconstrained mean-variance framework and come out with optimal portfolio weights that take extreme values (either negative or positive) in some assets. Green and Hollifield (1992) state that: *«[…] The extreme weights in efficient portfolios are due to the dominance of a single factor in the covariance structure of returns, and the consequent high correlation between naively diversified portfolios. With small amounts of cross-sectional diversity in asset betas, well-diversified portfolios can be constructed on subsets of the assets with very little residual risk and different betas. A portfolio of these diversified portfolios can then be constructed that has zero beta, thus eliminating the factor risk as well as the residual risk»*. This portfolio is unfeasible in practice and, unjustifiably, gets compared with observed investments in terms of Sharpe ratios[1]. This way, we relate actual investments with unrealistic ones, which ensure an even better performance than the optimal feasible portfolios. Hence, the comparison is erroneous since it tends to overestimate the inefficiency of any observed investment.

The problem is dealt with in Basak et al. (BJS, 2002) and Bucciol (2003); using a mean-variance approach, these authors develop an efficiency test in which the discriminating measure is based on a comparison no longer between Sharpe ratios, but between variances, for a given expected return. Such a technique, nevertheless, circumvents the above mentioned problem at the cost of neglecting some information: it just fixes the value of the expected return, and does not take into account how it could affect the importance of deviations in risk.

In this paper we try, then, to cope with inequality and equality constraints in a model that pays attention to expected returns as well as variance of investment returns. In lieu of working with efficient frontiers, we concentrate on the expected utility paradigm. Quoting Gourieroux and Monfort (2005), *«the main arguments for adopting the mean-variance approach and the normality assumption for portfolio management and statistical inference are weak and mainly based on their simplicity of implementation»*. It is well known (Campbell and Viceira, 2002) that from a myopic perspective the two procedures obtain the same results under several assumptions; the expected utility model, however, is more general and provides an economically more appealing measure of performance. Already Brennan and Torous (1999), Das and Uppal (2004) and Gourieroux and Monfort (2005) consider an agent who

[1] Any portfolio is indeed proportional to the zero-beta portfolio since the two fund separation theorem holds.

maximizes her expected utility in order to get an optimal portfolio. Brennan and Torous (1999), in particular, define a performance measure, based on the concept of compensative variation, which compares the utility from an optimal investment with that resulting from a given investment. Drawing inspiration from this strand of literature we will subsequently show that, using a power utility function, this procedure boils down to maximizing a function of mean and variance of a portfolio, for a given risk aversion; furthermore, the measure of compensative variation has the intuitive economic interpretation of the amount of wealth wasted or generated by the investment, relative to the optimal portfolio in one time-horizon.

The contribution of this paper is twofold. First, we characterize the asymptotic probability distribution and confidence intervals of the estimate of the wealth loss, and study its small sample distribution; this will permit us to perform statistically valid inference, and therefore to test for the efficiency of an investment. Already Gourieroux and Monfort (2005) introduce a test that accounts for market frictions. Our test based on the compensative variation metric, however, provides an economically more appealing statistic than a test based on the distance between portfolio weights. Second, we develop a strategy to estimate the implicit risk aversion parameter. Its estimate is the one that minimizes the wealth loss between the observed and the optimal alternative investment or, in other words, the one that makes the individual as much rational as possible.

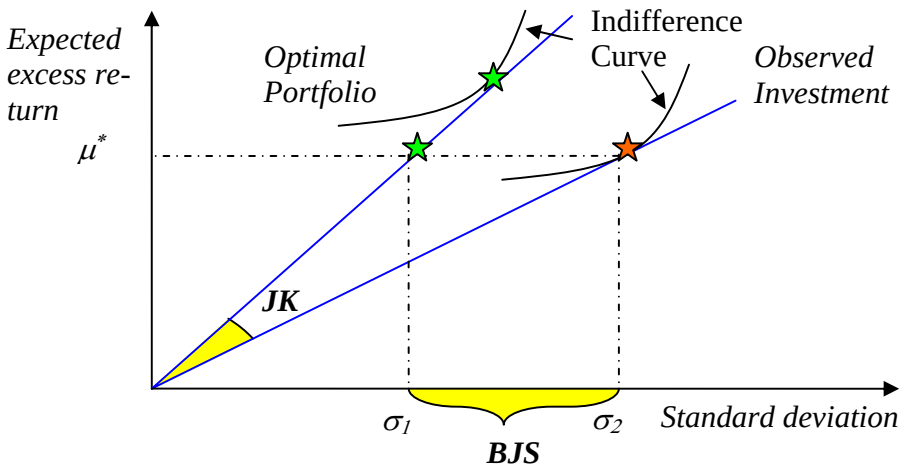
The paper is organized as follows: section 2 compares the standard mean-variance approach with our method based on expected utility maximization. It shows the underlying algebra of the agent's problem, and introduces a measure of wealth compensative variation. Section 3 describes the efficiency test; section 4 presents the statistic in a closed-form expression when there are no inequality constraints, and examines analogies with optimal portfolios derived in a mean-variance framework. Section 5 illustrates a powerful way to implicitly estimate the relative risk aversion parameter. As for the optimal portfolio weights, the expression can be derived in a clear closed-form expression only in the absence of inequality constraints; otherwise a solution can be obtained numerically. In section 6 we describe the data used in the empirical exercise, monthly returns for a value-weighted portfolio – that we call FF investment – and a set of ten average value-weighted industry portfolios from Fama and French and representative of the U.S. market. We further run some tests to assess the efficiency of the FF investment and a 1/n portfolio; we also compute the implicit risk aversion parameters associated with these strategies. In section 7 we run a block-bootstrap simulation to study the empirical distribution of our test. Section 8 summarizes the results and concludes; lastly, the appendix provides some mathematical details.

2. Agent's behavior

Disregarding constraints, we may assess the efficiency of an investment in a standard mean-variance framework, by comparing its Sharpe ratio with the optimal, as shown in figure 1[2]. It is the case, for instance, of the test proposed by JK in a portfolio setting. The optimal Sharpe ratio depicts the slope of the efficient frontier which includes a risk free asset within the endowment. The greater the difference between the two ratios, the greater the inefficiency of the observed investment (figure 1).

Figure 1.

Measures of efficiency – mean-variance framework



When constraints are included in the analysis, however, the Sharpe ratio is no longer an adequate metric to assess the efficiency. Some other tests, such as BJS, fix the level of expected return μ^* and consider the difference between σ_1^2 (the lowest achievable variance) and σ_2^2 (the observed variance). The smaller this difference, the lower the inefficiency of the observed investment. A caveat of this approach is that one dimension of the problem, the expected excess return, is kept fixed and therefore completely neglected by the efficiency analysis. Remaining in the mean-variance framework it is however difficult to think of different ways to face this problem, since there is no closed-form representation of the efficient frontier in the presence of inequality constraints.

A reasonable alternative is to consider an expected utility framework instead of a mean-variance approach. It is well known that the two methods are equivalent in a myopic perspective and under several assumptions; Campbell and Viceira (2002), for instance, argue that a power (or CRRA) utility function

[2] Although in the figure we draw an optimal portfolio with the same expected excess return as the observed investment, there are infinite optimal portfolios with the same Sharpe ratio; they differ only in the share invested in the risk free asset.

and log-normally distributed asset returns produce results consistent with a standard mean-variance analysis. The property of constant relative risk aversion, moreover, is attractive and helps explain the stability of financial variables over time.

We then draw inspiration from Gourieroux and Monfort (2005) and study the economic behavior of a rational agent who maximizes her expected utility of future wealth; such approach is appropriate even when the return distribution is not normal. We draw in figure 1 the indifference curves for an observed investment and the optimal portfolio. The optimal portfolio does not necessarily coincide with the one in the mean-variance metric, but in the absence of constraints it differs only for a scale factor (see §4). Our test considers the distance between the two indifference curves; the greater the distance, the greater the inefficiency.

Brennan and Torous (1999) analyze the same problem in a portfolio choice framework with a power utility function and come up with a measure of compensative variation that calculates the amount of wealth wasted when adopting a suboptimal portfolio allocation strategy; the same concept is used in Das and Uppal (2004) when assessing the relevance of systemic risk in portfolio choice, and in DeMiguel et al. (2006), among other methods, when measuring the inefficiency of an equally-weighted investment strategy. In the following sections we show how this measure of compensative variation can be used to develop an efficiency test whose validity is not affected by the presence of equality and/or inequality constraints on the portfolio asset shares.

2.1. An approach based on utility comparison

Following Brennan and Torous (1999), an investor maximizes at time t the expected value of a power utility function defined over her wealth at the end of the next period $t+dt$:

$$U\left(W_{t+dt}\right)=\frac{W_{t+dt}^{1-\gamma}-1}{1-\gamma}$$

where $\gamma>0$ is the relative risk aversion (RRA) coefficient and W_{t+dt} the wealth at time $t+dt$.

Our agent holds an investment b [3]. We assume that the price P_t^b at time t of the investment follows the stochastic differential equation

(1) $$\frac{dP_t^b}{P_t^b}=\mu_b dt+\sigma_b d\beta_t^b=\left(\eta_b+r_0\right)dt+\sigma_b d\beta_t^b$$

[3] The investment is generated from a different set of primitive assets than the optimal alternative investment. It could be a mutual fund, a pension fund, a market index, or a standard against which to measure the performance. BJS call this asset *benchmark*.

where $\mu_b = \eta_b + r_0$ (expected return) and σ_b (standard deviation) are constants, and $d\beta_t^b$ is the increment to a univariate Wiener process. The overall wealth W_t evolves with P_t^b:

$$\frac{dW_t}{W_t} = \frac{dP_t^b}{P_t^b}.$$

Using a property of the geometric Brownian motion, equation (1) implies that, over any finite interval of time $[t, t+dt]$

$$W_{t+dt} = W_t e^{\left(\mu_b - \frac{1}{2}\sigma_b^2\right)(t+dt-t) + \sigma_b\left(\beta_{t+dt}^b - \beta_t^b\right)} = W_t e^{\left(\mu_b - \frac{1}{2}\sigma_b^2\right)dt + \sigma_b\left(\beta_{t+dt}^b - \beta_t^b\right)}$$

with $\beta_t^b \sim N(0,t)$. In turn this implies that W_{t+dt} is log-normally distributed conditional to W_t, with expectation

$$E\left[W_{t+dt} \mid W_t\right] = W_t e^{\left(\mu_b - \frac{1}{2}\sigma_b^2\right)dt} E\left[e^{\sigma_b\left(\beta_{t+dt}^b - \beta_t^b\right)}\right] =$$

$$= W_t e^{\left(\mu_b - \frac{1}{2}\sigma_b^2\right)dt} e^{\frac{1}{2}\sigma_b^2(t+dt-t)} = W_t e^{\mu_b dt} = W_t e^{(\eta_b + r_0)dt}.$$

Therefore, the expected utility associated with the investment is given by

$$E\left[U\left(W_{t+dt}\right) \mid \eta_b, \sigma_b, \gamma, W_t\right] = \frac{1}{1-\gamma}\left(W_t^{1-\gamma} e^{(1-\gamma)\left(\eta_b + r_0 - \frac{1}{2}\gamma\sigma_b^2\right)dt} - 1\right).$$

In order to study the efficiency of this investment, an investor compares its performance with that of the optimal alternative: a portfolio of primitive assets. Without loss of generality, the endowment is given by one risk free asset (with return r_0) and a set of n risky assets (with return $r_i, i = 1,...,n$). Calling w_i the fraction of wealth allocated to the *i*-eth risky asset, w the vector of w_i's and $(1 - w'\iota)$ the residual fraction invested in the risk free asset, the overall wealth evolves as

$$\frac{dW_t}{W_t} = \left(w'\left(\mu_p - r_0\iota\right) + r_0\right)dt + \left(w'\Sigma_p w\right)^{1/2} d\beta_t =$$

$$= \left(w'\eta_p + r_0\right)dt + \left(w'\Sigma_p w\right)^{1/2} d\beta_t$$

where $d\beta_t$ is the increment to a univariate Wiener process, and $\mu_p = \eta_p + r_0\iota$ and Σ_p are the vector of the expected returns and the covariance matrix of the primitive assets.

Following the computation already made for the investment case, the expected utility is

$$E\left[U\left(W_{t+1}\right) \mid \eta_p, \Sigma_p, \gamma, W_t\right] = \frac{1}{1-\gamma}\left(W_t^{1-\gamma} e^{(1-\gamma)\left(\left(w'\eta_p + r_0\right) - \frac{1}{2}\gamma w'\Sigma_p w\right)dt} - 1\right).$$

We consider a "buy & hold" strategy in which the investor observes the asset returns at time t and makes her choice once and forever; it is intended to represent the type of inefficiency in portfolio allocations induced by the *status quo* bias described in Samuelson and Zeckhauser (1988).

The optimal portfolio w^* is defined as

(2) $$w^* = \arg\max_w E\left[U\left(W_{t+1}\right) | \eta_p, \Sigma_p, \gamma, W_t\right]$$

subject to several constraints (equality, inequality, sum to one etc.) on its composition:

$$Aw = a$$

$$lb \le w \le ub$$

A natural way to assess the performance of the investment, then, is to compare its expected utility with that resulting from the optimal alternative investment. In accordance with Brennan and Torous (1999) and Das and Uppal (2004), we establish this comparison in terms of compensative variation. In other words, we pose the question of what level of initial wealth $W_t^* = W_t\left(1 - cv_b\right)$ is needed to obtain with the optimal portfolio the same expected utility as with the investment and initial wealth W_t.

In formulae, we impose that

$$E\left[U\left(W_{t+1}\right) | \eta_p, \Sigma_p, \gamma, W_t\left(1 - cv_b\right)\right] = E\left[U\left(W_{t+1}\right) | \eta_b, \sigma_b, \gamma, W_t\right].$$

Solving for the compensative variation cv_b,

$$cv_b\left(\eta_b, \sigma_b^2, \eta_p, \Sigma_p, \gamma\right) = \left[1 - \exp\left\{\left(\eta_b - \frac{1}{2}\gamma\sigma_b^2\right)dt - \left(w^{*\prime}\eta_p - \frac{1}{2}\gamma w^{*\prime}\Sigma_p w^*\right)dt\right\}\right]$$

with $cv_b \in \left(-\infty, 1\right]$[4]. This function has a clear economic interpretation: it measures the fraction of initial wealth that an agent wastes (if positive) or generates (if negative) over the period $\left[t, t + dt\right]$, when investing in b instead of the best alternative. $cv_b = 1$ means that the investment is completely inefficient (the agent is wasting 100 percent of her wealth); $cv_b \to -\infty$, instead, means that the investment is totally efficient (the agent is generating infinite new wealth).

If we want to assess the efficiency of a portfolio ω, instead of an investment, against the optimal portfolio w^*, it is easy to show that the compensative variation is

$$cv_p = cv\left(\eta_p, \Sigma_p, \gamma\right) = \left[1 - \exp\left\{\left(\omega'\eta_p - \frac{1}{2}\gamma\omega'\Sigma_p\omega\right)dt - \left(w^{*\prime}\eta_p - \frac{1}{2}\gamma w^{*\prime}\Sigma_p w^*\right)dt\right\}\right]$$

[4] Note that the presence of a risk free asset in the endowment set is not essential, since its return r_0 disappears in cv_b. When a risk free asset is absent, an additional constraint $w'\iota = 1$ has to be imposed on the weights of the optimal portfolio.

where now $cv_p \in [0,1]$ since the observed portfolio ω comes from the *same* space of primitive assets as the optimal portfolio w^*. When $cv_p = 0$ the agent is investing in a portfolio that does not waste any wealth; it is, in other words, efficient.

We are able to associate to cv_b and cv_p a standard error, a confidence interval and an efficiency test; this will be shown in the next section. Before proceeding with the algebra, we define the utility loss λ_b as:

$$\lambda_b\left(\eta_b, \sigma_b^2, \eta_p, \Sigma_p, \gamma\right) = -\frac{1}{dt}\log\left(1 - cv_b\right) = \\ = \left(w^{*\prime}\eta_p - \frac{1}{2}\gamma w^{*\prime}\Sigma_p w^*\right) - \left(\eta_b - \frac{1}{2}\gamma\sigma_b^2\right) \tag{3}$$

in the case of an investment, and λ_p for a portfolio likewise; since $cv = 1 - \exp\{-dt\lambda\}$, maximizing the expected utility function or the utility loss provides the same results.

Below we ignore the constant term that involves dt [5], for the sake of simplicity and since it disappears when computing the test statistic.

3. Development of an efficiency test

The function $\lambda_b\left(\eta_b, \sigma_b^2, \eta_p, \Sigma_p, \gamma\right)$ depends on unknown moments[6] and is replaced with a consistent sample estimate, defined as

$$\ell_b = \ell_b\left(e_b, s_b^2, e_p, S_p, \gamma\right) = \max_w \left\{\left(w' e_p - \frac{1}{2}\gamma w' S_p w\right) - \left(e_b - \frac{1}{2}\gamma s_b^2\right)\right\} \tag{4}$$

$$\text{subject to } Aw = a$$

$$lb \le w \le ub$$

We solve, therefore, the maximization problem using a function of *sample* moments instead of *true* moments. As a consequence, we need to take account of sampling errors and derive a statistical distribution for the function ℓ_b. Yet deriving its exact distribution is tough and worthless. It is tough because the presence of inequality constraints hinders the recourse to standard statistical procedures; it is worthless as, even if we knew the exact distribution, in the end it would be a mixture of different distribu-

[5] The reader can assume that $dt = 1$.
[6] Let us assume for now that the relative risk aversion coefficient γ is known.

tions[7]. Therefore, even if we computed the exact distribution, this could be used only through numerical simulation. It would in fact be exactly the same procedure we should follow in the case of not knowing the exact distribution of ℓ_b.

Another possibility is to approximate the exact distribution by means of the delta method. Following BJS we use a weak version of the central limit theorem to establish that the first and second moments of returns are asymptotically normally distributed; we then calculate the derivative of ℓ_b relative to $\left(e_b, s_b^2, e_p, S_p\right)$, obtaining a first-order approximation of the exact distribution of $\ell_b\left(e_b, s_b^2, e_p, S_p, \gamma\right)$. The procedure is described in detail below.

First of all, we recognize that the only source of randomness in $\ell_b\left(e_b, s_b^2, e_p, S_p, \gamma\right)$ is given by the non-central first and second moments of the primitive assets and the investment. Since working with vectors is more convenient than working with matrices, once we define

$$e_b = \frac{1}{T}\sum_{t=1}^{T} e_{bt}\ ; \qquad M_b = \frac{1}{T}\sum_{t=1}^{T} M_{bt} = \frac{1}{T}\sum_{t=1}^{T} e_{bt}^2 = s_b^2 + e_b^2$$

$$e_p = \frac{1}{T}\sum_{t=1}^{T} e_{pt} = \frac{1}{T}\sum_{t=1}^{T}\begin{bmatrix} e_{p1t} \\ \vdots \\ e_{pnt} \end{bmatrix}; \qquad M_p = S_p + e_p e_p' = \frac{1}{T}\sum_{t=1}^{T} M_{pt} = \frac{1}{T}\sum_{t=1}^{T} e_{pt} e_{pt}'$$

we consider the vector $\overline{X}_T$ built as

$$\overline{X}_T = \begin{bmatrix} \overline{e}_T \\ \overline{M}_T \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} e_p \\ e_b \end{bmatrix} \\ \begin{bmatrix} vech\left(M_p\right) \\ M_b \end{bmatrix} \end{bmatrix} = \frac{1}{T}\sum_{t=1}^{T} X_t = \frac{1}{T}\sum_{t=1}^{T}\begin{bmatrix} e_t \\ M_t \end{bmatrix} = \frac{1}{T}\sum_{t=1}^{T}\begin{bmatrix} \begin{bmatrix} e_{pt} \\ e_{bt} \end{bmatrix} \\ \begin{bmatrix} vech\left(M_{pt}\right) \\ M_{bt} \end{bmatrix} \end{bmatrix} \tag{5}$$

where the operator *vech* takes all the distinct elements in a symmetric matrix:

$$vech\left(e_{pt} e_{pt}'\right) = \left[e_{p1t}^2,\ \ e_{p2t}e_{p1t},\ \ \cdots\ \ e_{pnt}e_{p1t},\ \ e_{p2t}^2,\ \ \cdots\ \ e_{pnt}e_{p2t},\ \ \cdots\ \ e_{pnt}^2\right]'.$$

It is worth stressing one more time that the investment returns come from a different, although possibly correlated, parametric space than the primitive asset returns. As a consequence the investment can be either more or less efficient than the portfolio.

[7] De Roon et al. (2001), testing for mean-variance spanning with short-sales constraints and transaction costs, obtain that their statistic is asymptotically distributed as a mixture of chi-squared distributions.

We require $\{X_t, t \geq 1\}$ to be a sequence of stationary and ergodic random vectors with mean $E[X_t] = X = \begin{bmatrix} \eta \\ M \end{bmatrix}$ and covariance matrix $\operatorname{cov}(X_t) = \Lambda$ with Λ non-singular. The expected value on $\overline{X}_T$ is, therefore, $E\left[\overline{X}_T\right] = X$ and its variance is

$$Var\left(\overline{X}_T\right) = Var\left(\frac{1}{T}\sum_{t=1}^{T} X_t\right) = \frac{1}{T}\left(\Lambda + \sum_{t=1}^{T-1}\left(1-\frac{t}{T}\right)\left(Cov(X_t, X_0) + Cov(X_0, X_t)\right)\right)$$

from which the long-run covariance matrix Λ_0 is

$$\Lambda_0 = \lim_{T\to\infty} T\ Var\left(\overline{X}_T\right) = \Lambda + 2\sum_{t=1}^{\infty} Cov(X_t, X_0).$$

Note that we do not exclude *a priori* the possibility of non-zero correlation between the investment and the primitive asset returns. Since the investment comes from a different parametric space than the primitive assets we just exclude *a priori* a perfect correlation (± 1) between the investment and the portfolio. The investment, in other words, can only be *partially* tracked by a portfolio.

Requiring some further mild assumptions (see §A.1 for details), we obtain that

$$\sqrt{T}\left(\ell\left(e_b, s_b^2, e_p, S_p, \gamma\right) - \lambda\left(\eta_b, \sigma_b^2, \eta_p, \Sigma_p, \gamma\right)\right) \xrightarrow{d} N(0, V)$$

with $V = \nabla(\gamma)' \Lambda_0 \nabla(\gamma)$, where

$$\nabla(\gamma) = \frac{\partial \lambda_b\left(\eta_b, \sigma_b^2, \eta_p, \Sigma_p, \gamma\right)}{\partial X} = \frac{\partial f(X, \gamma)}{\partial X}.$$

Define $\lambda\left(w, \delta \mid \overline{X}_T\right)$ as the Lagrangian and $\delta = [\delta_1 \quad \delta_2 \quad \delta_3]$ as the set of Lagrange multipliers:

$$\lambda\left(w, \delta \mid \overline{X}_T\right) = \left(w'e_p - \frac{1}{2}\gamma w' S_p w\right) - \left(e_b - \frac{1}{2}\gamma s_b^2\right) +$$
$$-\delta_1'(Aw - a) - \delta_2'(l - w) - \delta_3'(w - u).$$

By making use of the envelope theorem, the gradient $\nabla(\gamma)$ is consistently estimated by

$$D(\gamma) = \frac{\partial f\left(\overline{X}_T, \gamma\right)}{\partial \overline{X}_T} = \left.\frac{\partial \lambda\left(w, \delta \mid \overline{X}_T\right)}{\partial \overline{X}_T}\right|_{\substack{w=w^* \\ \delta=\delta^*}}.$$

The derivative is worth

$$D(\gamma)' = \left[\left(w^* + \left(\gamma e_p' w^*\right) w^*\right)', \quad -1 - \gamma e_b, \quad -\frac{1}{2}\gamma\left[w_1^{*2} \quad 2w_1^* w_2^* \quad \cdots \quad 2w_1^* w_n^* \quad w_2^{*2} \quad \cdots \quad 2w_2^* w_n^* \quad \cdots \quad w_n^{*2}\right], \quad \frac{1}{2}\gamma\right].$$

Lastly, we replace Λ_0 with its standard heteroskedasticity and autocorrelation consistent estimate L_0 as proposed by Newey and West (1987) and make use of Bartlett-type weights:

$$L_0 = \hat{\Omega}_0 + \sum_{j=1}^{m}\left(1-\frac{j}{m+1}\right)(\hat{\Omega}_j + \hat{\Omega}_j{}')$$

with

$$\hat{\Omega}_j = \frac{1}{T}\sum_{t=j+1}^{T}\left(X_t - \overline{X}_T\right)\left(X_{t-j} - \overline{X}_T\right)'$$

and m the number of lags to be considered.

The test statistic is then

(6) $$t = T^{1/2}\frac{\ell_b - \lambda_b}{\hat{V}^{1/2}} = T^{1/2}\frac{\ell_b - \lambda_b}{\left(D(\gamma)' L_0 D(\gamma)\right)^{1/2}}.$$

Under the null hypothesis $H_0 : \lambda_b = \lambda_0$, $t \overset{a}{\sim} N(0,1)$. Notice that the null can be equivalently written as $H_0 : cv_b = cv_0 = 1 - \exp\{-\lambda_0\}$. This second specification highlights, however, a shortcoming of our procedure: since $cv_b \in (-\infty, 1]$, we are not able to test whether $cv_b = 1$. A similar issue arises in Snedecor and Cochran (1989), when trying to test a null hypothesis of a variance equal to zero. In their framework, a statistic with an exact distribution exists for any value of the variance, except when the variance is zero, i.e., on the boundary of the feasible set[8]. In our framework testing the hypothesis $H_0 : cv_b = 1$ is not economically relevant: it is indeed hard to imagine an investment, however badly managed, able to dissipate *all* the wealth. We can however test any other hypothesis, and in particular if $cv_b = 0$, that is, if the investment can perfectly replicate the performance of the optimal portfolio.

Given the large sample distribution for $\ell_b\left(e_b, s_b^2, e_p, S_p, \gamma\right)$, a $(1-\alpha)$-confidence interval for λ_b is derived by

$$\alpha = \Pr\left(z_{\frac{\alpha}{2}} \le T^{1/2}\frac{\ell_b - \lambda_0}{\hat{V}^{1/2}} \le z_{1-\frac{\alpha}{2}}\right) = \Pr\left(\ell_b - z_{1-\frac{\alpha}{2}}\frac{\hat{V}^{1/2}}{T^{1/2}} \le \lambda_0 \le \ell_b + z_{1-\frac{\alpha}{2}}\frac{\hat{V}^{1/2}}{T^{1/2}}\right)$$

where $z_{1-\frac{\alpha}{2}}$ is the $\left(1-\frac{\alpha}{2}\right)$-eth percentile of a standard normal distribution.

[8] An analogous situation is reported in Kim et al. (2005) when dealing with Sharpe-style regressions. The asymptotic distribution of the style coefficients is valid only under the assumption that none of the true style weights are zero or one. In practice, it seems to be quite plausible to have zero or one as the values of some style weights.

Since $cv_0 = 1 - \exp\{-\lambda_0\}$, a confidence interval for the wealth loss is

$$CI(cv_0) = \left\{ cv_b : cv_b \in \left[1 - \exp\left\{ -\left(\ell_b - z_{1-\frac{\alpha}{2}} \frac{\hat{V}^{1/2}}{T^{1/2}} \right) \right\}, 1 - \exp\left\{ -\left(\ell_b + z_{1-\frac{\alpha}{2}} \frac{\hat{V}^{1/2}}{T^{1/2}} \right) \right\} \right] \right\}.$$

If we are instead interested in testing the efficiency of a portfolio, namely, an investment resulting from the same set of primitive assets as the optimal alternative, once we define

$$\overline{X}_T = \frac{1}{T} \sum_{t=1}^{T} X_t = \begin{bmatrix} e_p \\ vech(M_p) \end{bmatrix} = \frac{1}{T} \sum_{t=1}^{T} \begin{bmatrix} e_{pt} \\ vech(M_{pt}) \end{bmatrix}$$

it is straightforward to see that

$$D(\gamma)' = \left[\left(w^* + \left(\gamma e_p' w^* \right) w^* \right)', \quad -\frac{1}{2}\gamma \left[w_1^{*2}, \quad 2w_1^* w_2^*, \quad \cdots \quad 2w_1^* w_n^*, \quad w_2^{*2}, \quad \cdots \quad 2w_2^* w_n^*, \quad \cdots \quad w_n^{*2} \right] \right] +$$
$$-\left[\left(\omega + \left(\gamma e_p' \omega \right) \omega \right)', \quad -\frac{1}{2}\gamma \left[\omega_1^2, \quad 2\omega_1\omega_2, \quad \cdots \quad 2\omega_1\omega_n, \quad \omega_2^2, \quad \cdots \quad 2\omega_2\omega_n, \quad \cdots \quad \omega_n^2 \right] \right]$$

and that a confidence interval for the wealth loss cv_0 is

$$CI(cv_0) = \left\{ cv_p : cv_p \in \left[1 - \exp\left\{ -\left(\ell_p - z_{1-\frac{\alpha}{2}} \frac{\hat{V}^{1/2}}{T^{1/2}} \right) \right\}, 1 - \exp\left\{ -\left(\ell_p + z_{1-\frac{\alpha}{2}} \frac{\hat{V}^{1/2}}{T^{1/2}} \right) \right\} \right] \right\}.$$

This specification of the test does not hold true for cv_p equal to either 0 or 1; to investigate if $cv_p = 0$ ($cv_p = 1$) we then look at the confidence interval to cv_0 and check if its lower (upper) boundary is equal to zero (one).

4. Closed-form solutions with no inequality constraints

The expression of the test derived in §3 does not explicitly state the optimal portfolios. We are in fact able to derive their closed-form expression only in the absence of inequality constraints, namely, when i) there are no constraints or ii) there are only equality constraints. Otherwise we have to rely on numerical solutions; for instance, a Matlab® code which implements the function `quadprog` can efficiently solve the problem numerically.

Below we consider separately the two cases referring to the efficiency of an investment; analogous results apply for a portfolio. We also prove the existence of a strong relationship between standard mean-variance and expected utility maximization paradigms.

4.1. No constraints

We call $\ell^{NO}\left(e_b, s_b^2, e_p, S_p, \gamma\right)$ the difference between utilities in the absence of constraints:

$$\ell^{NO}\left(e_b, s_b^2, e_p, S_p, \gamma\right) = \max_w \left\{ \left(w' e_p - \frac{1}{2}\gamma w' S_p w \right) - \left(e_b - \frac{1}{2}\gamma s_b^2 \right) \right\}.$$

In this case the optimal weights are

(7) $$w_{NO}^* = \frac{1}{\gamma} S_p^{-1} e_p .$$

Replacing expression (7) into $\ell^{NO}\left(e_b, s_b^2, e_p, S_p, \gamma\right)$ we have then

$$\ell^{NO}\left(e_b, s_b^2, e_p, S_p, \gamma\right) = \frac{1}{2\gamma} e_p{}' S_p^{-1} e_p - \left(e_b - \frac{1}{2}\gamma s_b^2 \right).$$

We recognize in equation (7) an expression similar to the optimal weights in the standard unconstrained mean-variance analysis, where the optimal portfolio can be any of the infinite ones with the highest Sharpe ratio. The weights of the optimal portfolio with the same excess return r_b as the investment are then given by

(8) $$w_{NO}^{BJS} = \frac{r_b S_p^{-1} e_p}{e_p' S_p^{-1} e_p} .$$

This expression identifies the optimal portfolio used in BJS, where an agent aims at minimizing the variance of her investment given the expected return r_b.

It can be shown that, when we impose that the portfolio weights sum to one then:

- the Sharpe ratio of the portfolio (8) is equivalent to the Sharpe ratio of the tangency portfolio (TP),

$$w_{TP}^{BJS} = \frac{S_p^{-1} e_p}{\iota' S_p^{-1} e_p} ;$$

- the optimal portfolio that maximizes the expected utility of the agent with the highest optimal expected utility is the tangency portfolio, i.e., exactly the same portfolio we have in a standard mean-variance setting.

The optimal portfolio resulting in our expected utility framework in the case of no constraint is: i) equivalent to the optimal one in the mean-variance framework if there are no risk free assets and ii) is

otherwise proportional. Both equations (7) and (8) share indeed the same numerator $S_p^{-1}e_p$; the different denominators just normalize the weights. In other words, the importance of the two quantities

$$\frac{e_p' S_p^{-1} e_p}{r_b}; \qquad \gamma$$

is in establishing the fraction of wealth to invest in the risk free asset; the relationship within risky shares is instead kept fixed. This implies that the two portfolios lie on the same efficient frontier; see for instance the two optimal portfolios in figure 1. According to the two fund separation theorem, they could be seen as a combination of the tangency risky portfolio and a risk free asset.

4.2. Equality constraints only

If, instead, we define the function $\ell^{EQ}\left(e_b, s_b^2, e_p, S_p, \gamma\right)$ that takes account of equality constraints on some of the optimal portfolio weights,

$$\ell^{EQ}\left(e_b, s_b^2, e_p, S_p, \gamma\right) = \max_w \left\{ \left(w'e_p - \frac{1}{2}\gamma w' S_p w \right) - \left(e_b - \frac{1}{2}\gamma s_b^2 \right) \right\}$$

subject to $Aw = a$

the Lagrangian is

$$\lambda\left(w, \delta_1 \mid \overline{X}_T\right) = w'e_p - \frac{1}{2}\gamma w' S_p - we_b + \frac{1}{2}\gamma s_b^2 - \delta_1'\left(Aw - a\right)$$

with δ_1 Lagrange multiplier relative to the equality constraints.

Taking the derivative with respect to w and δ_1,

(9) $$\begin{cases} w = \dfrac{1}{\gamma} S_p^{-1}\left(e_p + A'\delta_1\right) \\ Aw = a \end{cases}$$

we solve for δ_1 to get

$$Aw = a = \frac{1}{\gamma} A S_p^{-1}\left(e_p + A'\delta_1\right) \Rightarrow \delta_1^* = \left(A S_p^{-1} A'\right)^{-1}\left(\gamma a - A S_p^{-1} e_p\right)$$

from which

$$\begin{aligned} w_{EQ}^* &= \frac{1}{\gamma} S_p^{-1}\left(e_p + A'\delta_1^*\right) = \\ &= \frac{1}{\gamma}\left(I - S_p^{-1}A'\left(A S_p^{-1} A'\right)^{-1} A\right) S_p^{-1} e_p + S_p^{-1} A'\left(A S_p^{-1} A'\right)^{-1} a = \frac{1}{\gamma} Q + q. \end{aligned}$$

Replacing this expression in the objective function,

$$\ell^{EQ}\left(e_b, s_b^2, e_p, S_p, \gamma\right) = \left(\frac{1}{\gamma}Q'e_p + q'e_p - \frac{1}{2\gamma}Q'S_pQ - \frac{1}{2}Q'S_pq - \frac{1}{2}q'S_pQ - \frac{1}{2}\gamma q'S_pq\right) - \left(e_b - \frac{1}{2}\gamma s_b^2\right).$$

In order to make a comparison with the existing literature, we follow GJ and split the primitive assets in two groups:

$$w = \begin{bmatrix} \tilde{w}_1 \\ \tilde{w}_2 \end{bmatrix}; \qquad e_p = \begin{bmatrix} \tilde{e}_1 \\ \tilde{e}_2 \end{bmatrix}; \qquad S_p = \begin{bmatrix} S_{11} & S_{12} \\ S_{12}' & S_{22} \end{bmatrix}$$

and impose the constraint $\tilde{w}_2 = \tilde{\omega}_2$.

After some algebra we obtain

(10) $$\tilde{w}_1^* = \frac{1}{\gamma}S_{11}^{-1}\tilde{e}_1 - S_{11}^{-1}S_{12}\tilde{\omega}_2 .$$

In selecting the optimal values, an agent has then to take into account a hedge term against the constrained assets. Dealing with equality constraints allows one to model the presence of transaction costs in some assets that, for this reason, are illiquid. For instance, using Italian data and the GJ test, Pelizzon and Weber (2003) observe that housing is a relevant component (nearly 80%) of the overall wealth of Italian households, and the efficiency greatly improves when this real asset is taken as a fixed component of the overall portfolio. Bucciol (2003) bears out their results and shows that the efficiency improves further when also inequality constraints are taken into account.

In a setting à la GJ, we are given the optimal portfolio

$$w_{EQ}^{GJ} = w_{EQ}^{BJS} = \begin{cases} \dfrac{\left(r_b - \tilde{\omega}_2'\left(\tilde{e}_2 - S_{12}'S_{11}^{-1}\tilde{e}_1\right)\right)}{\tilde{e}_1'S_{11}^{-1}\tilde{e}_1}S_{11}^{-1}\tilde{e}_1 - S_{11}^{-1}S_{12}\tilde{\omega}_2 \\ \tilde{\omega}_2 \end{cases}$$

where r_b is the expected excess return on the observed portfolio. Given the expected return r_b the optimal portfolio is exactly the same when computed with the BJS test.

Moreover, with the restriction that the weights sum to one,

$$w_{TP}^{BJS} = \begin{cases} \left(\dfrac{1 - \iota'\tilde{\omega}_2 + \iota'S_{11}^{-1}S_{12}\tilde{\omega}_2}{\iota'S_{11}^{-1}\tilde{e}_1}\right)S_{11}^{-1}\tilde{e}_1 - S_{11}^{-1}S_{12}\tilde{\omega}_2 \\ \tilde{\omega}_2 \end{cases}.$$

In our framework, instead, extending equation (10) to all the primitive assets, the optimal portfolio is given by

$$w_{EQ}^{*} = \begin{cases} \frac{1}{\gamma} S_{11}^{-1}\tilde{e}_1 - S_{11}^{-1}S_{12}\tilde{\omega}_2 \\ \tilde{\omega}_2 \end{cases}$$

and, if we require the sum to one,

$$w_{EQ}^{**} = \begin{cases} \left(\frac{1 - \iota'\tilde{\omega}_2 + \iota' S_{11}^{-1} S_{12}\tilde{\omega}_2}{\iota' S_{11}^{-1}\tilde{e}_1} \right) S_{11}^{-1}\tilde{e}_1 - S_{11}^{-1}S_{12}\tilde{\omega}_2 \\ \tilde{\omega}_2 \end{cases}$$

i.e., exactly the same equation obtained in a setting à la BJS. Without imposing the sum to one, the only difference with the GJ and BJS tests is, as before, in the normalization term: on the one hand, we have the expression

$$\frac{\left(r_b - \tilde{\omega}_2' \left(\tilde{e}_2 - S_{12}' S_{11}^{-1}\tilde{e}_1 \right) \right)}{\tilde{e}_1' S_{11}^{-1}\tilde{e}_1}$$

whereas, on the other, we have only the term γ. The same remarks made in §4.1 apply here.

In summary, despite slight differences the optimal behavior in an expected utility model with no inequality constraints is similar to the mean-variance framework. If we add inequality constraints, instead, we do not have any closed-form solution for the optimal portfolios, and therefore we are not able to make any analytical comparison.

5. The implicit relative risk aversion coefficient

The knowledge of the relative risk aversion coefficient γ is crucial since it is decisive in determining the size of investment in risky assets, as we see for example in equation (7).

By definition, γ depends neither on time nor on wealth:

$$\gamma = -W_t \frac{U''(W_t)}{U'(W_t)} .$$

It is well known, however, (see Stutzer, 2004, for a review) that its exact value for an investor is as hard to know as it is to estimate it through *ad hoc* questions. Rabin and Thaler (2001) believe that any method used to measure a coefficient of relative risk aversion is doomed to failure, since *«the correct conclusion for economists to draw, both from thought experiments and from actual data, is that people do not display a consistent coefficient of relative risk aversion, so it is a waste of time to try to measure it»*.

In this section we show how to obtain an estimate of the relative risk aversion parameter γ within this expected utility framework. Since an investor's portfolio is in principle informative on her risk aversion, it is reasonable to impute γ from the observed portfolio. Calvet et al. (2006), for instance, estimate the RRA coefficient directly from the optimal portfolio formula, assuming the investor is aware of the standard deviation and the Sharpe ratio of her portfolio. Our procedure is however more closely related to that in Gourieroux and Monfort (2005); they test the efficiency of a portfolio using a statistic which depends on an exogenous preference parameter. Should the parameter not *a priori* be given, they obtain a reasonable estimate of it by minimizing the statistic with respect to it. In our setting, the role of the preference parameter is played by γ. By solving a similar problem for our objective function (ℓ_b or ℓ_p), we estimate the implicit risk aversion as the one for which the welfare loss is minimized. Under the hypothesis that the portfolio is managed in such a way that the expected utility function is maximized, the estimator $\hat{\gamma}$ provides a consistent estimate for the RRA parameter.

It is straightforward to develop a procedure for deriving $\hat{\gamma}$ in a portfolio setting. Since the function $\ell_p\left(e_p, S_p, \gamma\right)$ is always non-negative, we can estimate γ by choosing the value that makes the objective function as small as possible, i.e., leads to the lowest inefficiency. In formulae, we solve

$$\hat{\gamma} = \arg\min_{\gamma} \max_{w} \left\{ \left(w'e_p - \frac{1}{2}\gamma w' S_p w \right) - \left(\omega' e_p - \frac{1}{2}\gamma \omega' S_p \omega \right) \right\}$$

subject to several constraints.

5.1. No constraints

If there are no restrictions, the estimate of γ is given by

$$\hat{\gamma}_{NO} = \arg\min_{\gamma} \left\{ \frac{1}{2\gamma} e_p{}' S_p^{-1} e_p + \frac{1}{2}\gamma \omega' S_p \omega - \omega' e_p \right\}.$$

It leads us to

$$\hat{\gamma}_{NO} = \left(\frac{e_p{}' S_p^{-1} e_p}{\omega' S_p \omega} \right)^{1/2}. \tag{11}$$

Knowing the analytical expression, we can also derive a standard error and a confidence interval for $\hat{\gamma}_{NO}$, following the same steps as in §3 (see §A.2).

5.2. Other constraints

Analogously, in the presence of just equality constraints, we solve the problem

$$\hat{\gamma}_{EQ} = \arg\min_{\gamma} \left\{ \left(\frac{1}{\gamma} Q'e_p + q'e_p - \frac{1}{2\gamma} Q'S_p Q - \frac{1}{2} Q'S_p q - \frac{1}{2} q'S_p Q - \frac{1}{2} \gamma q'S_p q \right) - \left(\omega' e_p - \frac{1}{2} \gamma \omega' S_p \omega \right) \right\}.$$

Deriving with respect to γ,

$$\hat{\gamma}_{EQ} = \left(\frac{2Q'e_p - Q'S_p Q}{\omega' S_p \omega - q'S_p q} \right)^{1/2} \tag{12}$$

although a solution among the positive numbers not always exists.

When inequality constraints are also present, it is no longer possible to find a closed-form expression for the implicit risk aversion parameter γ; we know, nevertheless, that the function

$$\min_{\gamma} \ell_p \left(e_p, S_p, \gamma \right) = \min_{\gamma} \max_{w} \left\{ \left(w'e_p - \frac{1}{2} \gamma w' S_p w \right) - \left(\omega' e_p - \frac{1}{2} \gamma \omega' S_p \omega \right) \right\}$$

determines the first order condition

$$\frac{\partial \ell_p \left(e_p, S_p, \gamma \right)}{\partial \gamma} = \left. \frac{\partial \lambda \left(e_p, S_p, \delta \right)}{\partial \gamma} \right|_{w=w^*} = \frac{1}{2} \omega' S_p \omega - \frac{1}{2} w^* \left(\gamma \right)' S_p w^* \left(\gamma \right) = 0 .$$

The optimal γ is therefore implicitly defined by the equation

$$\omega' S_p \omega = w^* \left(\gamma \right)' S_p w^* \left(\gamma \right).$$

In the investment case we can look at the value that maximizes the objective function, namely, the value for which the investment is as much efficient as possible. Absent inequality constraints, the formulae read close to (11) and (12):

$$\hat{\gamma}_{NO} = \left(\frac{e'_p S_p^{-1} e_p}{s_b^2} \right)^{1/2} ; \quad \hat{\gamma}_{EQ} = \left(\frac{2Q'e_p - Q'S_p Q}{s_b^2 - q'S_p q} \right)^{1/2} .$$

We can derive a standard error for $\hat{\gamma}_{NO}$, mimicking the procedure used in a portfolio setting (see §A.3).

6. Empirical analysis

For illustrative purpose we perform two separate analyses on the efficiency of an investment and a portfolio. As investment we consider a value-weighted portfolio constructed using market equity and NYSE breakpoints for a set of NYSE, AMEX, and NASDAQ stocks. Only the third quintile of such stocks, ranked according to their size, is included in the index that we are going to call “FF investment”

hereafter, to emphasize that it has been derived by Fama and French[9]. As observed portfolio we consider a 1/n strategy to the set of primitive assets. In both of the cases our primitive assets are a set of ten average value-weighted industry portfolios representative of the U.S. market made available by Kenneth French. The industry is divided into non-durable, durable, manufacturing, energy, hi-tech, telecommunication, shops, health, utilities and other sectors. No risk free asset is included. The data are monthly returns covering the period February 1950 through May 2005 (664 observations).

Table 1 reports descriptive statistics for our sample; notice that the variance associated with the FF investment is slightly higher than that of the health industry portfolio, which also guarantees a slightly higher expected return. This asset therefore *dominates* the investment in a mean-variance sense.

Table 1.

Descriptive statistics for industry portfolios and FF investment returns

Expected return

%	NoDur	Durbl	Manuf	Enrgy	HiTec	Telcm	Shops	Hlth	Utils	Other	FF
	1.0836	1.0241	1.0144	1.1960	1.1993	0.9088	1.0342	1.1782	0.9312	1.0836	1.1685

Covariance (normal) and correlation (italic) of percentage returns

%	NoDur	Durbl	Manuf	Enrgy	HiTec	Telcm	Shops	Hlth	Utils	Other	FF
NoDur	17.892	*0.6417*	*0.8177*	*0.4945*	*0.5755*	*0.6272*	*0.8383*	*0.7498*	*0.6345*	*0.8237*	*0.7844*
Durbl	14.968	30.414	*0.7854*	*0.4641*	*0.6202*	*0.5702*	*0.7469*	*0.4918*	*0.4588*	*0.7511*	*0.7517*
Manuf	16.379	20.512	22.425	*0.6243*	*0.7415*	*0.6188*	*0.8243*	*0.7257*	*0.5484*	*0.8933*	*0.8819*
Enrgy	10.578	12.943	14.948	25.569	*0.4192*	*0.3906*	*0.4498*	*0.4484*	*0.5459*	*0.6041*	*0.5764*
HiTec	15.791	22.185	22.777	13.751	42.075	*0.5974*	*0.6874*	*0.6359*	*0.3159*	*0.7107*	*0.7893*
Telcm	11.334	13.434	12.518	8.4369	16.555	18.250	*0.6568*	*0.5412*	*0.5326*	*0.6732*	*0.6433*
Shops	17.389	20.201	19.142	11.153	21.866	13.759	24.049	*0.6601*	*0.5101*	*0.8400*	*0.8414*
Hlth	15.757	13.475	17.072	11.263	20.491	11.487	16.082	24.681	*0.4791*	*0.7191*	*0.6551*
Utils	10.262	9.6734	9.9285	10.554	7.8355	8.6987	9.5639	9.1005	14.618	*0.6120*	*0.5577*
Other	17.037	20.256	20.687	14.939	22.543	14.064	20.144	17.471	11.443	23.914	*0.8878*
FF	16.627	20.773	20.927	14.606	25.657	13.771	20.676	16.310	10.685	21.756	25.112

Note: NoDur=Non-Durable, Durbl=Durable, Manuf=Manufacturing, Enrgy=Energy, HiTec=Hi-Tech, Telcm=Telecommunication, Hlth=Health, Utils=Utilities, FF= Fama and French index formed on market equity value.

Using these data, we compute the optimal portfolios for our t test with different levels of risk aversion, imposing that the weights sum to one and diverse further constraints (none, equality constraints, short-sales constraints, equality and short-sales constraints). As equality constraints, we require that

$$\begin{cases} w_{Hlth} & = & 0.1 \\ w_{Enrgy} + w_{Utils} & = & 0.2 \end{cases}$$

to represent a commitment to invest a fixed amount of wealth in "socially useful" industries (health, energy and public utility industries), disregarding their performance. This choice of the equality constraints has three further motivations: this way i) we prevent a complete investment in the asset (health)

[9] Downloadable from Kenneth French's website: http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html.

dominating the FF investment, ii) we impose restrictions that are not respected by the weights in the unconstrained optimal portfolios (see table 2), and iii) we set constraints compatible with a 1/n strategy.

In table 2 we report the composition of the 1/n portfolio, also known in the literature as equally-weighted or naïve portfolio, and that of several optimal portfolios, under the mean-variance (MV), or expected utility frameworks, considering different constraints and risk aversion levels. In each portfolio the weights sum to one since no risk free asset is available. It is worth pointing out that, when we refer to the unconstrained case, we in fact mean that one equality constraint (the sum to one of the weights) holds.

Without inequality constraints, the optimal portfolios hold several extreme positions, often short (2 to 4). Such portfolios provide the best performance, but are typically unfeasible in the reality, and comparing them with real investments is misleading. By imposing short-sales restrictions, the optimal portfolios are formed from only small subsets of assets; four primitive assets in particular (durable, manufacturing, shops, and other sectors) are never in the investment decisions. Not surprisingly, these are the assets that offer the lowest return/risk profiles, or that show the highest correlation with other assets. With the combination of all restrictions, the equality constraint on the sum of the weights associated with the energy and utilities sectors is responsible for a null position on the utilities sector when γ is small.

The last five columns in table 2 summarize expected return and standard deviation of the optimal portfolio, its Sharpe ratio (SR), and the utility loss resulting from investing in the optimal alternative portfolio instead of the FF investment (ℓ_b) or the naïve portfolio (ℓ_p). Given the set of constraints, when γ increases in the power utility function, the expected return and the standard deviation of optimal portfolios decline, but the direction of the Sharpe ratio and the utility loss is unclear. The addition of constraints does not necessarily imply that the optimal portfolio in an expected utility framework is less efficient in a Sharpe ratio sense; the inclusion of short-sales constraints, indeed, increases SR. But remind that, since the optimal portfolios are derived from the minimization of the utility loss, the relevant measure of efficiency is ℓ_b or ℓ_p instead of SR. Coherently, we observe that the utility loss is constantly smaller.

Table 2.

Observed and optimal portfolios

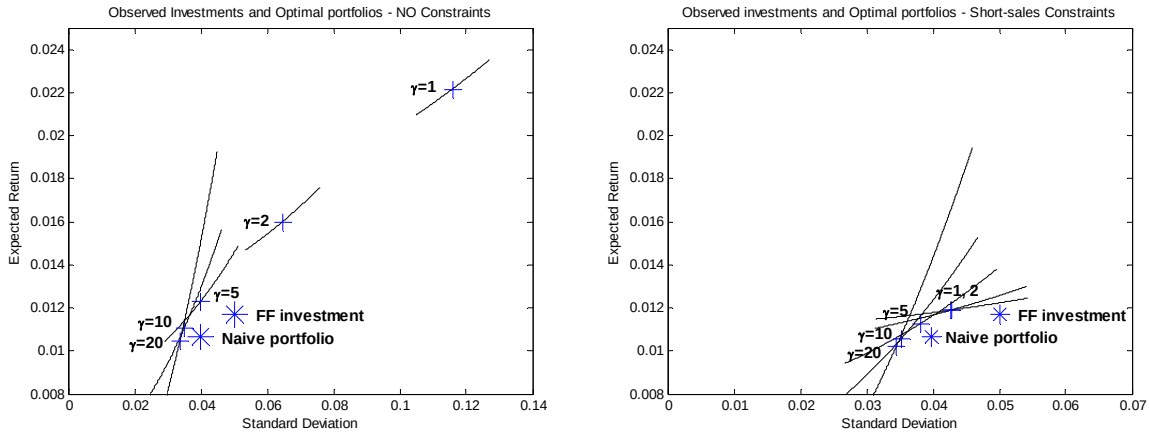
%	NoDur	Durbl	Manuf	Enrgy	HiTec	Telcm	Shops	Hlth	Utils	Other	Exp. Ret.	Std. Dev.	Sharpe ratio (SR)	Utility Loss (ℓ_b)	Utility Loss (ℓ_p)
1/n	10	10	10	10	10	10	10	10	10	10	1.06	3.98	26.78	-	-
No constraints															
MV	48.59	14.84	-43.72	35.59	12.92	13.09	-3.49	23.81	25.77	-27.41	1.12	3.55	31.64	-	-
γ=1	267.22	73.06	-412.75	186.10	112.07	-127.83	-66.69	113.75	-122.61	77.70	2.22	11.58	19.13	0.90	0.56
γ=2	144.10	40.28	-204.94	101.35	56.24	-48.48	-31.10	63.10	-39.06	18.51	1.60	6.47	24.74	0.62	0.27
γ=5	70.23	20.61	-80.26	50.49	22.74	-0.86	-9.74	32.72	11.08	-17.00	1.23	3.99	30.80	0.53	0.16
γ=10	45.61	14.05	-38.70	33.54	11.57	15.01	-2.63	22.59	27.79	-28.84	1.11	3.50	31.62	0.62	0.22
γ=20	33.30	10.77	-17.92	25.07	5.99	22.94	0.93	17.53	36.14	-34.76	1.05	3.37	31.05	0.89	0.43
Equality constraints (Health= 10%, Energy + Utilities = 20%)															
γ=1	355.29	54.85	-373.22	173.01	126.20	-104.96	-93.01	10	-153.01	104.86	2.12	11.25	18.86	0.85	0.50
γ=2	199.01	31.86	-179.64	89.19	60.42	-29.56	-50.20	10	-69.19	38.12	1.55	6.40	24.23	0.58	0.23
γ=5	105.25	18.06	-63.48	38.91	20.95	15.67	-24.51	10	-18.91	-1.93	1.21	4.12	29.34	0.48	0.11
γ=10	73.99	13.47	-24.77	22.14	7.79	30.75	-15.95	10	-2.14	-15.28	1.09	3.68	29.74	0.54	0.14
γ=20	58.36	11.17	-5.41	13.76	1.21	38.29	-11.67	10	6.24	-21.95	1.04	3.56	29.12	0.75	0.29
Short –Sales constraints															
γ=1	0	0	0	52.18	10.95	0	0	36.87	0	0	1.19	4.28	27.77	0.46	0.11
γ=2	0	0	0	49.81	7.46	0	0	42.73	0	0	1.19	4.26	27.88	0.45	0.10
γ=5	23.43	0	0	33.92	3.59	0.01	0	24.27	14.78	0	1.13	3.81	29.56	0.46	0.09
γ=10	17.04	0	0	23.01	0	14.63	0	16.43	28.89	0	1.06	3.52	29.98	0.56	0.16
γ=20	13.51	0	0	17.17	0	20.96	0	11.64	36.71	0	1.02	3.45	29.63	0.81	0.35
Short –Sales and Equality constraints (Health= 10%, Energy + Utilities = 20%)															
γ=1	27.89	0	0	20	42.11	0	0	10	0	0	1.16	4.51	25.81	0.42	0.08
γ=2	48.30	0	0	20	21.70	0	0	10	0	0	1.14	4.10	27.82	0.42	0.07
γ=5	52.53	0	0	20	6.66	10.81	0	10	0	0	1.10	3.85	28.71	0.43	0.06
γ=10	42.72	0	0	16.69	0	27.28	0	10	3.31	0	1.06	3.67	28.86	0.51	0.11
γ=20	36.05	0.75	0	9.81	0	33.21	0	10	10.19	0	1.03	3.61	28.53	0.70	0.24

Note: NoDur=Non-Durable, Durbl=Durable, Manuf=Manufacturing, Enrgy=Energy, HiTec=Hi-Tech, Telcm=Telecommunication, Hlth=Health, Utils=Utilities; MV = Mean-Variance portfolio. The FF investment has a mean of 1.17%,, a standard deviation of 5.01%, and a Sharpe ratio of 23.32%.

In figure 2 we plot the optimal portfolios for the t test and their indifference curves against the FF investment and the naïve portfolio; our test compares the indifference curves of the investment and the optimal portfolio.

Figure 2.

Efficient portfolios and indifference curves in a mean-standard deviation plan



6.1. Investment case

We already know that the FF investment is dominated in a mean-variance sense by one of the primitive assets. But the test introduced in this paper does not constrain the expected return to be equal to that of the investment we are evaluating, and takes the RRA parameter γ as exogenous. We then expect that the rejection of the null might be related to the risk aversion. Were the investment only moderately sub-optimal, it might even be considered efficient for an investor with very low risk aversion.

In table 3 we show the result of a t test under the null hypothesis $H_0 : \lambda_b\left(\eta_b, \sigma_b^2, \eta_p, \Sigma_p, \gamma\right) = 0$ and using the automatic selection rule

$$m = \text{int}\left(4\left(\frac{T}{100}\right)^{2/9}\right),$$

as suggested by Newey and West (1994), for the number of lags in the HAC covariance estimator. From the table we conclude that the FF investment is always inefficient for very high levels of risk aversion ($\gamma \geq 10$) and always efficient otherwise. When $\gamma = 5$, however, the p-value is not far from the standard threshold of 5 percent. The wealth loss, that ranges between 0.26 and 0.50 percent with no constraints and for $\gamma \in \{1, 2, 5\}$, does not change significantly when adding equality constraints, or

when adding short-sales constraints and γ is high; it is, instead, much smaller when the risk aversion is small and short-sales constraints are included. In this case, it ranges between 0.05 and 0.22 percent when $\gamma \in \{1, 2, 5\}$, its standard error reduces considerably, and the p-value of the test gets higher.

Table 3.

Wealth loss and t test of efficiency – FF investment

%	γ=1	γ=2	γ=5	γ=10	γ=20	γ=1	γ=2	γ=5	γ=10	γ=20
	No Constraints					Equality Constraints				
Wealth Loss	0.5002	0.2635	0.2894	0.5777	1.2430	0.4445	0.2231	0.2421	0.5014	1.1029
Std. Error	0.4181	0.2288	0.1569	0.1699	0.2303	0.3891	0.2056	0.1361	0.1484	0.2038
Lower 95% Conf. Int.	-0.3227	-0.1859	-0.0186	0.2440	0.7906	-0.3210	-0.1806	-0.0249	0.2103	0.7026
Upper 95% Conf. Int.	1.3164	0.7109	0.5964	0.9102	1.6934	1.2042	0.6252	0.5084	0.7918	1.5015
Test*	1.1934	1.1504	1.8419	3.3894	5.3633	1.1400	1.0840	1.7775	3.3716	5.3816
P-Value	0.2327	0.2500	0.0655	0.0007	0	0.2543	0.2784	0.0755	0.0007	0
	Short –sales Constraints					Short –sales and Equality Constraints				
Wealth Loss	0.0549	0.0890	0.2213	0.5196	1.1652	0.0194	0.0548	0.1926	0.4697	1.0598
Std. Error	0.1256	0.1281	0.1231	0.1483	0.2093	0.0911	0.0931	0.1083	0.1331	0.1875
Lower 95% Conf. Int.	-0.1917	-0.1623	-0.0202	0.2286	0.7541	-0.1593	-0.1278	-0.0200	0.2086	0.6917
Upper 95% Conf. Int.	0.3008	0.3397	0.4623	0.8098	1.5746	0.1978	0.2371	0.4048	0.7302	1.4266
Test*	0.4366	0.6950	1.7962	3.4951	5.5344	0.2128	0.5884	1.7761	3.5217	5.6235
P-Value	0.6624	0.4871	0.0725	0.0005	0	0.8315	0.5562	0.0757	0.0004	0

* *Null hypothesis: wealth loss equal to zero.*
Note: Equality constraints: Health=10%, Energy + Utilities= 20%.

In table 4 we derive the implicit relative risk aversion coefficient; its point estimate is close to a reasonable 3.[10] The associated one-month wealth loss corresponds to a 0.23 percent, and a t test of efficiency does not reject the null.

Table 4.

Optimal RRA coefficient and statistics based on it – FF investment

No constraints	RRA	Wealth Loss (%)	Test*
Optimal Value	2.9611	0.2312	1.2798
Std. Error	0.9377	0.1804	-
Lower 95% Conf. Int.	1.1233	-0.1231	-
Upper 95% Conf. Int.	4.7989	0.5841	-
P-Value*	-	-	0.2006

* *Null hypothesis: wealth loss equal to zero.*

[10] It is equal to 2.9998 with equality constraints, 0.3927 with short-sales constraints and 0.6965 with short-sales and equality constraints. Gollier (2002) observes that levels of γ higher than 10 are implausible.

According to the t test, we conclude that the investment in the FF index is inefficient only for unrealistic levels of risk aversion; in any other case the wealth loss is not significantly different from zero. The inclusion of short-sales constraints, in particular, explains most of the wealth loss detected by a test run on an unconstrained portfolio.

6.2. Portfolio case

In this section we deal with naïve, equally-weighted strategies, and try to establish how costly they are. There are several reasons for studying a naïve portfolio. First, it is easy to implement because it requires neither to optimize – and is therefore accessible even to investors with difficulty in diversifying (DeMiguel et al., 2005) – nor to estimate the moments of the asset returns. Kallberg and Ziemba (1984), for instance, show that small changes in the input parameters may result in large changes in composition of the optimal portfolio. Second, it is empirically proven (Benartzi and Thaler, 2001) that investors often use such simple rules for allocating their wealth across assets.

Let us suppose that an agent follows a naïve strategy, i.e., invests exactly the same fraction 1/n of wealth in each of the n assets. Such portfolio is badly diversified and clearly inefficient under a mean-variance metric; a JK test run using the 10 industry portfolios produces indeed a p-value of 0.0403. Yet the literature does not provide clear evidence supporting the inefficiency of this simple strategy. Brennan and Torous (1999), for instance, infer from a simulation analysis that even an equally-weighted portfolio of as few as five randomly chosen firms can provide the same level of expected utility as the market portfolio. In DeMiguel et al. (2005) the strategy performs quite well too. The authors argue then that, even if naïve diversification results in a lower performance than optimal diversification, the loss is smaller than the one generated from the use of parameters estimated with error as inputs for the optimizing models. We therefore expect that our test, based on a measure different from the standard Sharpe ratios, and considering market frictions, concludes for a very small inefficiency of a naïve strategy.

Table 5 shows the wealth wasted when holding this portfolio instead of the optimal under several constraints. The wealth however wasted is smaller (below 0.2%) for γ between 5 and 10; it grows for more elevate and small risk aversion levels. The inclusion in the analysis of short-sales constraints, however, produces even in the case of $\gamma = 1$ a very small wealth loss. Given the lower boundary of the confidence interval, we do not have enough evidence to conclude that the naïve strategy is inefficient when $\gamma = 1$, 2 or 5.

In the same table we report the results of a t test in which the null hypothesis is that the wealth loss suffered by investing in a 1/n strategy is the same with and without constraints. Only when equality and short-sales restrictions are binding we find enough evidence to conclude that imposing these restrictions significantly reduces the inefficiency of the naïve strategy with respect to the case of absence of investment restrictions. This makes us believe that the effect of a combination of constraints may be stronger than the sum of the effects of single constraints because of their interrelations. The inclusion of equality constraints, in particular, does not result in a significantly smaller wealth loss. We thus argue that, for reasonable levels of risk aversion, according to this model it is not possible to conclude that a naïve strategy is inefficient, and that, as we add more constraints, the point estimate of its wealth loss decreases significantly for low risk-averse individuals. In this case, therefore, accounting for market frictions helps explain much of the rationale behind the recourse to this strategy.

Table 5.

Wealth loss and t test of efficiency – naïve portfolio

%	γ=1	γ=2	γ=5	γ=10	γ=20	γ=1	γ=2	γ=5	γ=10	γ=20
	No Constraints					Equality Constraints				
Wealth Loss	0.5568	0.2741	0.1616	0.2198	0.4280	0.5012	0.2337	0.1143	0.1433	0.2867
Std. Error	0.4050	0.2006	0.0996	0.0921	0.1182	0.3854	0.1869	0.0839	0.0714	0.0898
Lower 95% Conf. Int.	0	0	0	0.0391	0.1960	0	0	0	0.0032	0.1106
Upper 95% Conf. Int.	1.3475	0.6666	0.3569	0.4001	0.6595	1.2536	0.5995	0.2785	0.2831	0.4625
Test*	-	-	-	-	-	-0.1444	-0.2160	-0.5645	-1.0723	-1.5754
P-Value	-	-	-	-	-	0.8852	0.8290	0.5724	0.2836	0.1152
	Short –sales Constraints					Short – sales and Equality Constraints				
Wealth Loss	0.1117	0.0997	0.0935	0.1615	0.3496	0.0763	0.0654	0.0647	0.1114	0.2433
Std. Error	0.0779	0.0777	0.0590	0.0680	0.0972	0.0570	0.0387	0.0419	0.0510	0.0701
Lower 95% Conf. Int.	0	0	0	0.0281	0.1589	0	0	0	0.0115	0.1059
Upper 95% Conf. Int.	0.2643	0.2519	0.2091	0.2947	0.5399	0.1878	0.1412	0.1468	0.2112	0.3806
Test*	-5.7259	-2.2471	-1.1545	-0.8575	-0.8076	-8.4570	-5.4016	-2.3142	-2.1288	-2.6382
P-Value	0	0.0246	0.2483	0.3911	0.4193	0	0	0.0207	0.0333	0.0083

* *Null hypothesis: wealth loss equal with and without constraints.*
Note: Equality constraints: Health=10%, Energy + Utilities= 20%.

Lastly, we see in table 6 that the optimal γ derived using equation (11) takes a value of 5.0680[11] and however not higher than 7.3688 in a 95 percent confidence interval. In other words, the agent with the smallest wealth loss has a risk aversion roughly equal to 5. The corresponding wealth loss is about

[11] 5.7577 with the equality constraint, 3.8974 with short-sales constraints, and 3.9223 with short-sales and equality constraints.

0.16 percent in a one-month period, and its confidence interval produces a lower bound just equal to zero.

Table 6.

Optimal RRA coefficient and statistics based on it – naïve portfolio

NO CONSTRAINTS	RRA	WEALTH LOSS (%)
OPTIMAL VALUE	5.0680	0.1616
STD. ERROR	1.1739	0.0990
LOWER 95% CONF. INT.	2.7672	0
UPPER 95% CONF. INT.	7.3688	0.3554

In most cases, therefore, in our analysis we do not have enough information to reject the null of efficiency of a naïve strategy, and even the point estimate of the wealth loss is so small that it is actually comparable with, for instance, management fees in the mutual fund market, and, however, cheaper than any reasonable cost of information search. Consistently with the literature, many investors could thus prefer this solution to a theoretically more efficient portfolio.

7. Empirical distribution of the test

In this section we investigate the accuracy of the test asymptotic distribution in small samples. The statistic $\ell_b = \ell_b\left(e_b, s_b^2, e_p, S_p, \gamma\right)$ is a highly non-linear function of the random variables and for this reason the small sample distribution of the test may significantly differ from its normal asymptotic distribution. The knowledge of the test small sample properties may have relevant implication on the empirical analysis. For instance, Bucciol (2003) makes use of a statistic closely related to BJS; partly because of a small sample size, he concludes for the efficiency of most Italian household portfolios, but this efficiency seems to be too widespread to be explained just with the addition of inequality constraints.

To establish the small-sample properties of the t test we then perform a block-boostrap simulation (Kunsch, 1989). Given the time series of the observed data,

$$\left\{ e_t = \begin{bmatrix} e_{pt} \\ e_{bt} \end{bmatrix}, t = 1, \ldots, T \right\}$$

we adopt the following algorithm for the investment test:

1. Compute the sample moments e_p, e_b, S_p, s_b^2 and consequently the statistics $\ell_0 = \left(w^{*\prime} e_p - \frac{1}{2}\gamma w^{*\prime} S_p w^* \right) - \left(e_b - \frac{1}{2}\gamma s_b^2 \right)$, $cv_0 = 1 - \exp\{-\ell_0\}$, $V_0 = \hat{V}\left(e_b, s_b^2, e_p, S_p, \gamma\right)$;
2. Define the block size $b = \text{int}\left(T^{1/5}\right)$ according to Hall et al. (1995)[12]; the number of blocks is thus $k = \text{int}\left(\frac{T}{b}\right)$ and the length of any bootstrap sample is $L = kb \le T$;
3. Repeat the following a number N of times, with $j = 1, \ldots, N$:
   a. Generate a random i.i.d. sample $\{i_0, i_1, \ldots i_{k-1}\}$ from a discrete uniform distribution on $\{1, 2, \ldots, T - b + 1\}$;
   b. Construct a bootstrap pseudo-series $\{e_t^j, t = 1, \ldots, L\}$ as $e_{mb+h}^j = e_{i_m + h - 1}$ for $m = 0, \ldots, k-1$ and $h = 1, \ldots, b$;
   c. Compute the statistics $\ell^j = \left(w^{*\prime} e_p^j - \frac{1}{2}\gamma w^{*\prime} S_p^j w^* \right) - \left(e_b^j - \frac{1}{2}\gamma s_b^{2\,j} \right)$ and $cv^j = 1 - \exp\{-\ell^j\}$.

The block-bootstrap distribution of the statistic $\ell_b = \ell_b\left(e_b, s_b^2, e_p, S_p, \gamma\right)$ is given by the sample distribution of $L^{1/2}\left(\ell^j - \ell_0\right)$ for $j = 1, \ldots, N$. Under the null $H_0 : \ell_b = \ell_{H_0}$, we compare the bootstrap realizations with the statistic $T^{1/2}\left(\ell_0 - \ell_{H_0}\right)$; a similar algorithm is implemented for a portfolio test. In this paper we show results using $N = 1000$; different values of N do not provide significant differences, and neither does a diverse number of primitive assets (we tried with 5 and 30 instead of 10).

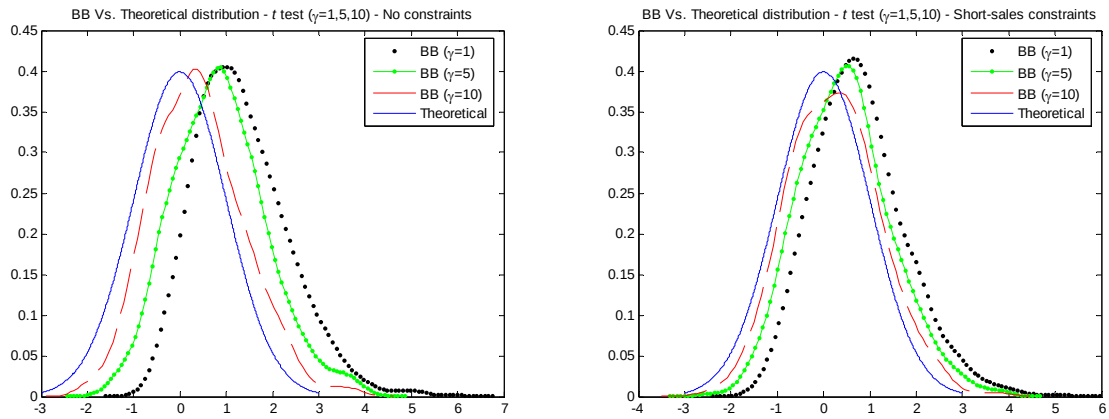
Figure 3 compares the theoretical (solid line) with the simulated (dashed line) distribution for the investment test when $\gamma = 1$, 5, 10 and no constraints (left panel) or short-sales constraints (right panel) are considered[13]; the test statistics have been rescaled according with their variance, V^{BB}. We notice that 1) the empirical statistic actually appears to be normally distributed, 2) the estimated variance adequately replicates the asymptotic variance, especially in the constrained case, but 3) the empirical distribution is not centered around zero and 4) the bias is larger as the risk aversion is smaller.

[12] In this case $b = 3$, $k = 221$ and $l = 663$. Hall et al. (1995) show that following this rule it is possible to determine the optimal block size in the case of estimation of a two-sided distribution function. The optimality is meant in terms of minimization of the mean square error of the block-bootstrap estimator.

[13] We do not report a figure with equality constraints because the inclusion in the optimization problem of such restrictions does not seem to alter the conclusions.

Figure 3.

Block-Bootstrap (BB) Vs. theoretical distribution – FF investment



The bias

$$bias = L^{1/2}\left(\frac{1}{N}\sum_{j=1}^{N}\ell^{j} - \ell_0\right)$$

is always higher than zero, with an average that tends to disappear as $L \to \infty$. Table 7 shows the average bias for an investment test with $\gamma = 5$, different time series lengths, and assuming normality in the asset returns.

Table 7.

Average bias (%), $\gamma = 5$ – FF investment

	$L = 500$	$L = 664$	$L = 1000$	$L = 1500$	$L = 2000$
No Constraints	4.2596	3.6388	3.1681	2.3647	2.2169
Equality Constraints	3.3293	2.7877	2.5056	1.8016	1.7732
Short –sales Constraints	1.6671	1.4807	1.3852	0.9847	0.9627
Short –sales and Equality Constraints	0.9943	0.8376	0.7702	0.5287	0.5096

Note: Equality constraints: Health=10%, Energy + Utilities= 20%.

With time series of length $L = 2000$ we get a bias roughly equal to half ($(500/2000)^{1/2}$) of the bias with $L = 500$. In a separate analysis, available upon request, we ran a block-bootstrap procedure to prove that the small sample bias depends on the application of the delta method to the highly non-linear function $\ell_b = \ell_b\left(e_b, s_b^2, e_p, S_p, \gamma\right)$. Similar conclusions are drawn from the analysis of a portfolio test, where in addition the simulated right tail seems to be longer than that of a normal distribution; see indeed figure 4 and table 8 for the case of a naïve portfolio.

Figure 4.

Block-Bootstrap (BB) Vs. theoretical distribution – naïve portfolio

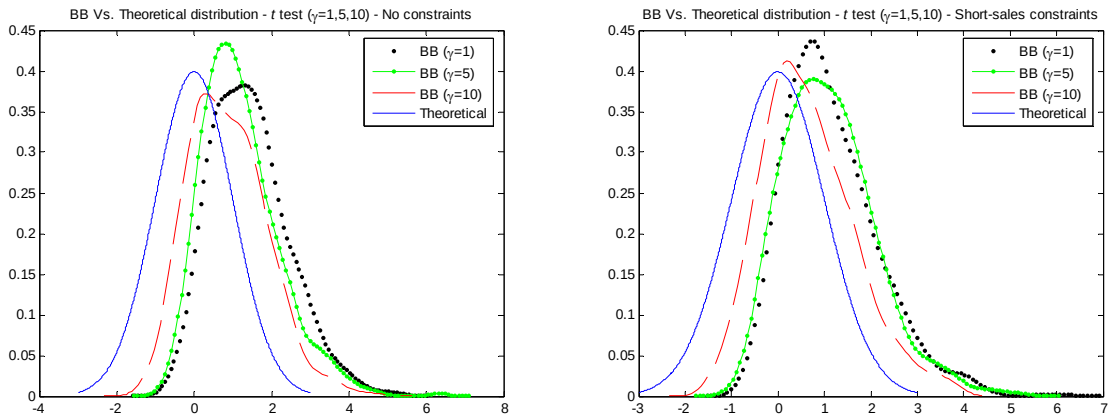


Table 8.

Average bias (%), $\gamma = 5$ – naïve portfolio

	$L = 500$	$L = 664$	$L = 1000$	$L = 1500$	$L = 2000$
NO CONSTRAINTS	4.2032	3.6577	2.9276	2.3572	2.0241
EQUALITY CONSTRAINTS	3.2445	2.8980	2.3195	1.8604	1.5154
SHORT –SALES CONSTRAINTS	1.6535	1.4842	1.2124	1.0692	0.9546
SHORT –SALES AND EQUALITY CONSTRAINTS	0.9799	0.8638	0.6857	0.5848	0.4702

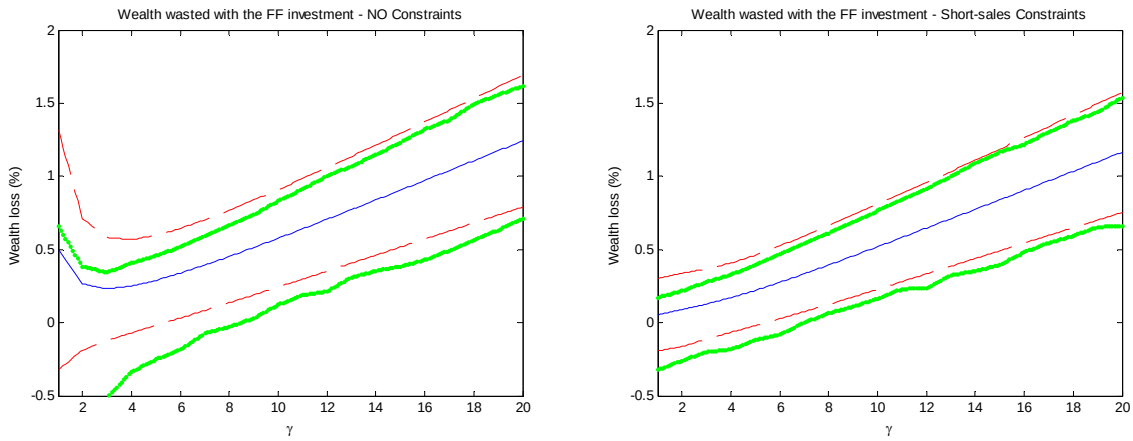
Note: Equality constraints: HEALTH=10%, ENERGY + UTILITIES= 20%.

The bias might, however, alter the conclusion of a hypothesis testing. We therefore repeat the analysis in §6 using the block-bootstrap distribution of the t test instead of the standard normal distribution. Figure 5 plots the amount of wealth wasted against the level of risk aversion, in the presence of no constraints (left panels) and only short-sales constraints (right panels) for the FF investment and the naïve strategy. When the risk aversion is reasonably small – $\gamma \leq 8$ for the FF investment, $\gamma \leq 13$ for the 1/n portfolio – we cannot reject the null of efficiency using the bootstrapped distribution of the wealth loss. The simulated 95% confidence intervals tend to approach the asymptotic ones as γ gets higher. The asymptotic distribution seems to work poorly in the absence of constraints and when γ is low.

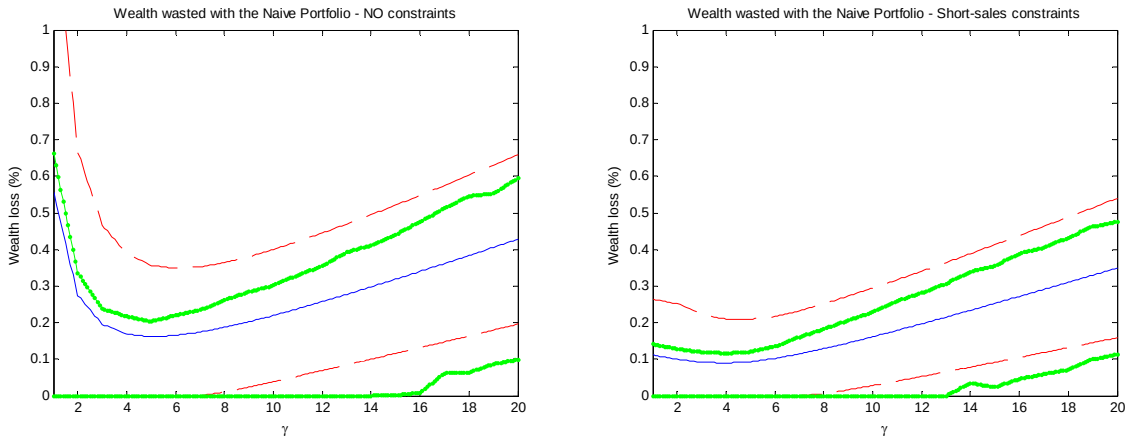
Figure 5.

Wealth loss (%) and confidence intervals

FF investment



Naïve portfolio



Note: *Wealth Loss,* *Asymptotic Conf. Int.,* *Block-Bootstrap Conf. Int.*

Table 9 reproduces (in italic) the values taken by the wealth loss and the test with the observed data discussed in §6. Using the block-bootstrap distribution, the table also reports the simulated standard error of the wealth loss and the rejection rates of the tests. The rejection rates are computed according to the formula

$$rr = \frac{2\min\left\{\#t^j < t_0, \#t^j > t_0\right\}}{N}$$

where $t^j, j = 1,\ldots,N$ is the block-bootstrap distribution of the test, and t_0 is the value of the test as observed in the original data.

Table 9.

Wealth loss and t test of efficiency based on the block-bootstrap distribution

FF investment

%	γ=1	γ=2	γ=5	γ=10	γ=20	γ=1	γ=2	γ=5	γ=10	γ=20
	No Constraints					Equality Constraints				
Wealth Loss	*0.5002*	*0.2635*	*0.2894*	*0.5777*	*1.2430*	*0.4445*	*0.2231*	*0.2421*	*0.5014*	*1.1029*
Std. Error	0.5345	0.2822	0.1768	0.1884	0.2411	0.4729	0.2581	0.1586	0.1605	0.2143
Lower 95% Conf. Int.	-1.4126	-0.7094	-0.2503	0.1226	0.7122	-1.2172	-0.6563	-0.2341	0.0819	0.6221
Upper 95% Conf. Int.	0.6552	0.3824	0.4581	0.8327	1.6171	0.6780	0.3053	0.3834	0.7265	1.4641
*Test**	*1.1934*	*1.1504*	*1.8419*	*3.3894*	*5.3633*	*1.1400*	*1.0840*	*1.7775*	*3.3716*	*5.3816*
Rej. Rate	0.7720	0.8540	0.4040	0.0220	0	0.9340	0.8120	0.4540	0.0180	0
	Short –sales Constraints					Short –sales and Equality Constraints				
Wealth Loss	*0.0549*	*0.0890*	*0.2213*	*0.5196*	*1.1652*	*0.0194*	*0.0548*	*0.1926*	*0.4697*	*1.0598*
Std. Error	0.1267	0.1219	0.1296	0.1556	0.2210	0.0937	0.0980	0.1080	0.1364	0.1887
Lower 95% Conf. Int.	-0.3233	-0.2608	-0.1184	0.1640	0.6600	-0.2419	-0.1985	-0.0601	0.1720	0.6477
Upper 95% Conf. Int.	0.1721	0.2164	0.3957	0.7647	1.5392	0.1400	0.1885	0.3705	0.6974	1.3765
*Test**	*0.4366*	*0.6950*	*1.7962*	*3.4951*	*5.5344*	*0.2128*	*0.5884*	*1.7761*	*3.5217*	*5.6235*
Rej. Rate	0.7900	0.9660	0.2240	0.0080	0	0.6880	0.9080	0.1400	0.0040	0

* *Null hypothesis: wealth loss equal to zero.*

Naïve portfolio

%	γ=1	γ=2	γ=5	γ=10	γ=20	γ=1	γ=2	γ=5	γ=10	γ=20
	No Constraints					Equality Constraints				
Wealth Loss	*0.5568*	*0.2741*	*0.1616*	*0.2198*	*0.4280*	*0.5012*	*0.2337*	*0.1143*	*0.1433*	*0.2867*
Std. Error	0.5146	0.2625	0.1285	0.0999	0.1266	0.4676	0.2496	0.1064	0.0818	0.0985
Lower 95% Conf. Int.	0	0	0	0	0.0981	0	0	0	0	0.0194
Upper 95% Conf. Int.	0.6621	0.3346	0.2033	0.3014	0.5944	0.6772	0.3071	0.1598	0.2058	0.4062
*Test**	-	-	-	-	-	*-0.1444*	*-0.2160*	*-0.5645*	*-1.0723*	*-1.5754*
Rej. Rate	-	-	-	-	-	0.1480	0.0900	0.0440	0.0280	0.0220
	Short –sales Constraints					Short –sales and Equality Constraints				
Wealth Loss	*0.1117*	*0.0997*	*0.0935*	*0.1615*	*0.3496*	*0.0763*	*0.0654*	*0.0647*	*0.1114*	*0.2433*
Std. Error	0.0820	0.0722	0.0577	0.0665	0.0970	0.0575	0.0458	0.0409	0.0506	0.0744
Lower 95% Conf. Int.	0	0	0	0	0.1130	0	0	0	0	0.0716
Upper 95% Conf. Int.	0.1408	0.1278	0.1203	0.2290	0.4759	0.0945	0.0839	0.0886	0.1637	0.3489
*Test**	*-5.7259*	*-2.2471*	*-1.1545*	*-0.8575*	*-0.8076*	*-8.4570*	*-5.4016*	*-2.3142*	*-2.1288*	*-2.6382*
Rej. Rate	0	0	0	0.0740	0.2300	0	0	0	0	0.0020

* *Null hypothesis: wealth loss equal with and without constraints.*

Note: In Italic values obtained from the observed statistic. Equality constraints: Health=10%, Energy + Utilities= 20%.

A comparison between simulated and asymptotic standard errors (see table 3 and 5) confirms that the asymptotic distribution works better in small samples as γ increases or short-sales constraints are included into the analysis. The magnitude of the rejection rates is, in some cases, very different from that of the p-values based on the theoretical distribution. In the case of the naïve strategy, the rejection rates suggest that, for small risk aversions, the wealth loss does not significantly decrease only when equality constraints are added. Comparing asymptotic p-values (see table 5) with simulated rejection

rates, in three cases (highlighted) with high values of γ and equality constraints we reverse our conclusion. Adding constraints to the standard case, therefore, seems to provide a significant reduction in the wealth loss, at least with small levels of risk aversion and for short-sales constraints. The overall impression is that, using the asymptotic distribution, we tend to reject too little the null hypothesis in the case of a naïve portfolio; this could explain the high fraction of efficient portfolios observed in Bucciol (2003).

Lastly, table 10 displays the simulated standard errors and confidence intervals for the optimal coefficient of risk aversion, in both the investment and portfolio case. The results are comparable with those in tables 4 and 6; we see higher standard errors, upward-shifted – but still reasonable – confidence intervals for the implicit risk aversion coefficient, but downward-shifted confidence intervals for the related wealth loss.

Table 10.

Optimal RRA coefficient and statistics based on it using the block-bootstrap distribution

No constraints	FF Investment			Naïve Portfolio	
	RRA	Wealth Loss (%)	Test*	RRA	Wealth Loss (%)
Optimal Value	*2.9611*	*0.2312*	*1.2798*	*5.0680*	*0.1616*
Std. Error	0.9601	0.1851	-	1.7331	0.1029
Lower 95% Conf. Int.	2.5148	-0.3614	-	4.2432	0
Upper 95% Conf. Int.	6.3133	0.3667	-	10.7775	0.2288
Rej. Rate*	-	-	0.7460	-	-

Note: In Italic values obtained from the observed statistic.

To summarize, although the asymptotic distribution is biased in small samples, the conclusions virtually keep the same using the block-bootstrap distribution with our data. In particular, we do not reject the efficiency of the FF index unless $\gamma > 8$, and we conclude for the efficiency of a naïve portfolio for risk aversions below 13. Including constraints into the analysis, furthermore, seems to decrease significantly the wealth loss, at least with risk aversion coefficients lower or equal to 10 and with short-sales constraints.

8. Concluding remarks

In this paper we study the efficiency of an investment in an expected utility framework, dealing with complex problems in which the optimal portfolio depends on weight constraints. We consider a measure of compensative variation which reads as the wealth loss between optimal and sub-optimal portfolios. We provide its asymptotic distribution and discuss a related efficiency test. The analysis of the

small sample properties of the test indicates that for standard time series lengths it is advisable to resort to the block-bootstrap (Kunsch, 1989) simulated distribution of the statistic.

We suggest an estimation strategy for the risk aversion parameter based on the level that minimizes the wealth loss with respect to the optimal portfolio. This estimate turns out to be useful when establishing the implicit risk aversion adopted, for instance, by fund managers when building their fund portfolio, or by households when choosing the composition of their portfolio. The statistic can flexibly deal with equality and inequality constraints on portfolio composition, even if the presence of inequality constraints makes it impossible to derive a closed-form solution. Although we depart from the classical literature of mean-variance analysis, we show to what extent the two frameworks are comparable and provide similar results.

Our illustrative empirical application provides examples of investments that are inefficient in the mean-variance metric, but for which our measure of wealth loss cannot reject the hypothesis that these sub-optimal strategies are indeed consistent with an optimizing behavior of an investor with a CRRA utility function and a reasonable level of risk aversion. In particular, our findings for the naïve 1/n investment strategy confirm the results in Brennan and Torous (1999) and Das and Uppal (2004). Considering short-sales constraints into the analysis, our point estimates of the wealth loss falls to about 0.10% per month.

In this paper we consider a short-term perspective in choosing the optimal portfolio. Yet it seems reasonable that many (institutional and individual) investors follow instead a long-term view when deciding how to allocate their wealth. Further research work will be devoted to extend our analysis to the behavior of forward-looking agents with regard to their lifetime portfolios.

A. Appendix

A.1. Assumptions and derivation of the asymptotic distribution of the test

In order to obtain the asymptotic distribution of the test we need a set of six mild assumptions; the first four are listed below.

(i) $\{X_t, t \geq 1\}$ is a sequence of stationary and ergodic random vectors with mean $E[X_t] = X = \begin{bmatrix} \eta \\ M \end{bmatrix}$ and covariance matrix $\operatorname{cov}(X_t) = \Lambda$ with Λ non-singular;

(ii) $\lim_{T\to\infty} E\left[|X_t|^{2+\delta}\right] < \infty$ and $\lim_{T\to\infty} Var(I'S_T) = \infty \quad \forall t \geq 1, \ \forall I \in \Re^n$ and $\forall \delta \in (0,1)$, where $S_T = \sum_{t=1}^{T} X_t$;

(iii) $\rho_I(t) = \lim_{t\to\infty} \max\{Corr(I'Y, I'Z)\} = 0 \quad \forall Y \in \sigma\{X_k : k \leq s\}, \ \forall Z \in \sigma\{X_k : k \geq t+s\}$ and $\forall I \in \Re^n$;

(iv) $\sum_{t=1}^{\infty} \|Cov(X_t, X_0)\| < \infty$ and $\Lambda_0 = \Lambda + 2\sum_{t=1}^{\infty} Cov(X_t, X_0)$ is non-singular.

Assumption (i) implies that $E\left[\overline{X}_T\right] = X$ and $\lim_{T\to\infty} T\ Var\left(\overline{X}_T\right) = \Lambda_0$. Assumption (ii) mimics the Lyapounov condition, and is required to achieve the uniform asymptotic negligibility condition of Lindeberg for the Central Limit Theorem to hold. The total variability of the sum, S_T, on the other hand, is always required to grow to infinity. Assumption (iii) ensures asymptotic independence; it is necessary to apply the Central Limit Theorem to non-i.i.d. random variables or random vectors. The first part of Assumption (iv), the finiteness condition, implies that Λ_0 exists and is finite, and that S_T in assumption (ii) grows at the same rate as T. The second part – the non-singularity of Λ_0 – is required to get a non-degenerate asymptotic distribution when applying the Central Limit Theorem. All these assumption are necessary to apply the result 1 in BJS (p. 1203) and identify a distribution for the vector $\overline{X}_T$:

$$\sqrt{T}\left(\overline{X}_T - X\right) \xrightarrow{d} N(0, \Lambda_0).$$

The second step is to obtain the asymptotic distribution of the statistic $\ell_b\left(e_b, s_b^2, e_p, S_p, \gamma\right) = f\left(\overline{X}_T, \gamma\right)$. In order to apply the delta method, the optimal solution λ_b in (3) has

to be a continuous function of the parameters, with a continuous first derivative in any point except for its boundaries. If so, it turns out that

$$\sqrt{T}\left(\ell_b\left(e_b, s_b^2, e_p, S_p, \gamma\right) - \lambda_b\left(\eta_b, \sigma_b^2, \eta_p, \Sigma_p, \gamma\right)\right) \xrightarrow{d} N\left(0, V\right)$$

with $V = \nabla\left(\gamma\right)' \Lambda_0 \nabla\left(\gamma\right)$, where

$$\nabla\left(\gamma\right) = \frac{\partial \lambda_b\left(\eta_b, \sigma_b^2, \eta_p, \Sigma_p, \gamma\right)}{\partial X} = \frac{\partial f\left(X, \gamma\right)}{\partial X}.$$

For this to be satisfied, whenever an inequality constraint is binding, the corresponding Lagrange multiplier has to be strictly positive. Following our previous notation we need, in other words, the sum of the inequality constraints and the Lagrange multipliers δ_2 and δ_3 to be strictly positive:

$$\left(w^* - lb\right) + \delta_2 > 0$$

$$\left(ub - w^*\right) + \delta_3 > 0$$

The following assumptions (v) and (vi) ensure that this is the case. To state them we need some additional notation. Let $\left\{i_1, \ldots, i_k, i_{k+1}, \ldots i_n\right\}$ be any permutation of $\left(1, \ldots, n\right)$. Thus $\left(\Sigma^{-1}\right)_2$ is the $\left(n-k\right) \times n$ matrix consisting of the $\left\{i_{k+1}, \ldots i_n\right\}$ rows of the Σ_p^{-1} and $\left(\Sigma^{-1}\right)_{22}$ is the $\left(n-k\right) \times \left(n-k\right)$ principal minor matrix which consists of $\left\{i_{k+1}, \ldots i_n\right\}$ rows and columns of the Σ_p^{-1}; lb_2 and ub_2 are the vectors consisting of the $\left\{i_{k+1}, \ldots i_n\right\}$ rows of lb and ub. Let us further define Σ_{11}^{-1} as the inverse of the $k \times k$ principal minor matrix corresponding to the $\left\{i_1, \ldots i_k\right\}$ rows and columns of the Σ_p.

We thus modify assumption (7) in BJS and require that

(v) the $\left(n-k\right)$-dimensional vector $\left(\left(\Sigma^{-1}\right)_{22}\right)^{-1}\left(\gamma lb_2 - \left(\Sigma^{-1}\right)_2\left(\eta_p - A'\delta_1\right)\right)$ is strictly positive in all its elements;

(vi) the $\left(n-k\right)$-dimensional vector $\left(\left(\Sigma^{-1}\right)_{22}\right)^{-1}\left(\left(\Sigma^{-1}\right)_2\left(\eta_p - A'\delta_1\right) - \gamma ub_2\right)$ is strictly positive in all its elements.

Assumption (v) is sufficient to ensure that $\left(w^* - lb\right) + \delta_2$ is a strictly positive vector. Note indeed that the first order condition to the maximization problem implies that

$$\gamma\Sigma_p w = \eta_p - A'\delta_1 + \delta_2 - \delta_3 .$$

Hence

$$w^* = \frac{1}{\gamma}\left(\Sigma_p^{-1}\left(\eta_p - A'\delta_1\right) + \Sigma_p^{-1}\left(\delta_2 - \delta_3\right)\right).$$

To see how assumption (v) works, suppose now that only some elements of w^* given by the sub-vector $\left[w_{i_{k+1}}^*, \ldots, w_{i_n}^*\right]$, are such that the constraint $w^* \geq lb$ holds with equality. We show that the corresponding sub-vector of Lagrange multipliers, $\left(\delta_{2i_{k+1}}, \ldots, \delta_{2i_n}\right)$ is positive. Without loss of generality assume that the last $(n-k)$ elements of w^* are such that the lower-inequality constraint is binding. Denote the vector of these elements as w_2^* and the rest as w_1^*, i.e., $w^* = \left[w_1^{*\prime} \quad w_2^{*\prime}\right]'$; therefore $w_2^* = lb_2$. Let δ_{21} and δ_{22}, δ_{31} and δ_{32} denote the corresponding partition of the Lagrange multiplier vector, i.e., $\delta_2 = \left[\delta_{21}' \quad \delta_{22}'\right]'$ and $\delta_3 = \left[\delta_{31}' \quad \delta_{32}'\right]'$. From the constraint $w^* - lb \geq 0$, δ_{21} is a zero vector. Furthermore, $\delta_{32} = 0$. Now partition Σ_p, Σ_p^{-1}, η_p and A, similarly, as

$$\Sigma_p = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}; \qquad \Sigma_p^{-1} = \begin{bmatrix} \Sigma^{11} & \Sigma^{12} \\ \Sigma^{21} & \Sigma^{22} \end{bmatrix}$$

$\eta_p = \left[\eta_1' \quad \eta_2'\right]'$, and $A = \left[A_1 \quad A_2\right]$, with Σ_{11} and Σ^{11} $k \times k$ matrices, η_1 and $A_1'\delta_1$ are $k \times 1$ vectors. This gives $w_2^* = \frac{1}{\gamma}\left(\left[\Sigma^{21} \quad \Sigma^{22}\right]\left(\eta_p - A'\delta_1\right) + \Sigma^{22}\delta_{22}\right) = lb_2$. Thus,

$$\delta_{22} = \left(\Sigma^{22}\right)^{-1}\left(\gamma lb_2 - \left[\Sigma^{21} \quad \Sigma^{22}\right]\left(\eta_p - A'\delta_1\right)\right) = \left(\left(\Sigma^{-1}\right)_{22}\right)^{-1}\left(\gamma lb_2 - \left(\Sigma^{-1}\right)_2\left(\eta_p - A'\delta_1\right)\right)$$

which is positive by the assumption.

Analogously, if assumption (vi) holds true,

$$\delta_{32} = \left(\Sigma^{22}\right)^{-1}\left(\left[\Sigma^{21} \quad \Sigma^{22}\right]\left(\eta_p - A'\delta_1\right) - \gamma ub_2\right) + \delta_{22} = \left(\Sigma^{22}\right)^{-1}\left(\left[\Sigma^{21} \quad \Sigma^{22}\right]\left(\eta_p - A'\delta_1\right) - \gamma ub_2\right) > 0 .$$

A.2. Confidence interval for the implicit risk aversion parameter relative to a portfolio

Let us define

$$\overline{Y}_T = \begin{bmatrix} e_p \\ vech\left(S_p\right) \end{bmatrix} = g\left(\begin{bmatrix} e_p \\ vech\left(S_p + e_p e_p'\right) \end{bmatrix}\right) = g\left(\overline{X}_T\right)$$

and $\hat{\gamma}_{NO} = h\left(\overline{Y}_T\right)$ with

$$V\left(\overline{X}_T\right) = \frac{\partial g\left(\overline{X}_T\right)}{\partial \overline{X}_T} = \begin{bmatrix} I_n & \underset{n\times\frac{n(n+1)}{2}}{\mathbf{0}} \\ \begin{bmatrix} G_1 \\ G_2 \\ \vdots \\ G_n \end{bmatrix} & I_{\frac{n(n+1)}{2}} \end{bmatrix}$$

and

$$G_i = -\begin{bmatrix} \underset{(n-i+1)\times(i-1)}{\mathbf{0}} & \begin{bmatrix} e_{pi} \\ e_{pi+1} \\ \vdots \\ e_{pn} \end{bmatrix} & \underset{(n-i+1)\times(n-i)}{\mathbf{0}} \end{bmatrix} - e_{pi}\begin{bmatrix} \underset{(n-i+1)\times(i-1)}{\mathbf{0}} & I_{n-i+1} \end{bmatrix}$$

Moreover,

$$Z\left(\overline{Y}_T\right)' = \left(\frac{\partial h\left(\overline{Y}_T\right)}{\partial \overline{Y}_T}\right)' = \begin{bmatrix} \underset{1\times n}{\mathbf{0}}, & \frac{1}{2\left(\omega' S_p \omega\right)^{1/2}\left(e_p' S_p^{-1} e_p\right)^{1/2}} OT \end{bmatrix} - \begin{bmatrix} \frac{e_p' S_p^{-1}}{\left(\omega' S_p \omega\right)^{1/2}\left(e_p' S_p^{-1} e_p\right)^{1/2}}, & \frac{\left(e_p' S_p^{-1} e_p\right)^{1/2}}{2\left(\omega' S_p \omega\right)^{3/2}} WT \end{bmatrix}$$

where

$$OT = \begin{bmatrix} \omega_1^2, & 2\omega_1\omega_2, & \cdots & 2\omega_1\omega_n, & \omega_2^2, & \cdots & 2\omega_2\omega_n, & \cdots & \omega_n^2 \end{bmatrix}$$

$$WT = \begin{bmatrix} e_p' DS_{11} e_p, & e_p' DS_{12} e_p, & \cdots & e_p' DS_{1n} e_p, & e_p' DS_{22} e_p, & \cdots & e_p' DS_{23} e_p, & \cdots & e_p' DS_{nn} e_p \end{bmatrix}$$

and DS_{ij} denotes the derivative of the (i, j)-eth element of S_p^{-1}.

Therefore, the standard error for $\hat{\gamma}_{NO}$ is

$$s.e.\left(\hat{\gamma}_{NO}\right) = \frac{1}{T^{1/2}}\left(Z\left(\overline{Y}_T\right)' V\left(\overline{X}_T\right) L_0 V\left(\overline{X}_T\right)' Z\left(\overline{Y}_T\right)\right)^{1/2}$$

and, applying the central limit theorem, a confidence interval for $\hat{\gamma}_{NO}$ is

$$CI\left(\hat{\gamma}_{NO}\right) = \left\{\gamma > 0 : \gamma \in \left[\hat{\gamma}_{NO} - z_{1-\frac{\alpha}{2}} s.e.\left(\hat{\gamma}_{NO}\right), \hat{\gamma}_{NO} + z_{1-\frac{\alpha}{2}} s.e.\left(\hat{\gamma}_{NO}\right)\right]\right\}.$$

A.3. Confidence interval for the implicit risk aversion parameter relative to an investment

What changes with respect to §A.2. is that

$$\overline{Y}_T = \begin{bmatrix} e_p \\ e_b \\ vech\left(S_p\right) \\ s_b^2 \end{bmatrix} = g\left(\overline{X}_T\right)$$

and $\hat{\gamma}_{NO} = h\left(\overline{Y}_T\right)$ with

$$V\left(\overline{X}_T\right) = \frac{\partial g\left(\overline{X}_T\right)}{\partial \overline{X}_T} = \begin{bmatrix} I_n & \underset{n\times 1}{\mathbf{0}} & \underset{n\times\frac{n(n+1)}{2}}{\mathbf{0}} & \underset{n\times 1}{\mathbf{0}} \\ \underset{1\times n}{\mathbf{0}} & 1 & \underset{1\times\frac{n(n+1)}{2}}{\mathbf{0}} & 0 \\ \begin{bmatrix} G_1 \\ \vdots \\ G_k \end{bmatrix} & \underset{\frac{n(n+1)}{2}\times 1}{\mathbf{0}} & I_{\frac{n(n+1)}{2}} & \underset{\frac{n(n+1)}{2}\times 1}{\mathbf{0}} \\ \underset{1\times n}{\mathbf{0}} & -2e_b & \underset{1\times\frac{n(n+1)}{2}}{\mathbf{0}} & 1 \end{bmatrix}$$

Finally,

$$Z\left(\overline{Y}_T\right) = \frac{\partial h\left(\overline{Y}_T\right)}{\partial \overline{Y}_T} = \begin{bmatrix} \dfrac{S_p^{-1}e_p}{\left(s_b^2\right)^{1/2}\left(e_p' S_p^{-1} e_p\right)^{1/2}} \\ 0 \\ \dfrac{1}{2\left(s_b^2\right)^{1/2}\left(e_p' S_p^{-1} e_p\right)^{1/2}} WT \\ -\dfrac{\left(e_p' S_p^{-1} e_p\right)^{1/2}}{2\left(s_b^2\right)^{3/2}} \end{bmatrix}.$$

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