Fractional Nonparametric Cointegration Analysis

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Abstract

This paper provides a theoretical fractional cointegration analysis in a nonparametric framework. We solve a generalized eigenvalues problem, extending Bierens' (1997) approach. To this end, a couple of random matrices is constructed, distinguishing between stationary and nonstationary part of the fractional integrated process. The asymptotic behavior of such matrices is studied and convergence results are obtained.

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1 Introduction

The concept of cointegration has been introduced by Granger(1981) and analyzed by Engel and Granger (1987). Most of the analyses have mainly considered the CI(1,1) cointegration case, in which two or more I(1) variables give rise to I(0) linear combinations and the long run relationships are derived with

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little or no restrictions on the short run dynamics. In order to avoid the knife-edge I(1)/I(0) distinction and to allow for potential slow adjustments towards long run equilibria, fractional cointegration approaches have been proposed. Consider a two-dimensional process $(X_t; Y_t)$ such that both variables are I(d) processes. We say that X_t and Y_t are fractionally cointegrated if there exists a linear combination $U_t = Y_t - BX_t$ such that U_t is $I(d_U)$, with $d_U < d$. Fractional cointegration is a generalization of standard cointegration, where d and d_U are 1 and 0, respectively. Parametric and semiparametric fractional cointegration models has focused on the reduction of the memory parameter from $d \geq \frac{1}{2}$ to $d_U < \frac{1}{2}$ since cointegration is commonly thought if as a stationary relationship between stationary, but cases in which the differencing parameter is less than $\frac{1}{2}$ are also discussed, in particular in the context of financial time series analysis. A partial list of works includes Jegathan (1999), Breitung and Hassler (2002), Davidson (2002), Robinson and Yajima (2002), Robinson and Hualde (2003), Nielsen (2004), Marmol and Velasco (2004). While Jegathan (1999), Breitung and Hassler (2002), Davidson (2002) and Robinson and Hualde (2003) developed parametric models, Robinson and Yajima (2002), Nielsen (2004), Marmol and Velasco (2004) and Robinson and Iacone (2005) worked in a semiparametric context.

In this paper new theoretical nonparametric approach is proposed. The contribution of this work to the literature on cointegration analysis is as follows: First, the nonparametric approach of Bierens (1997) is modified by defining new weight functions for the random matrix related to the nonstationary and stationary part of the fractional integrated process. Second, the asymptotic results for such random matrices are obtained by using functional analysis theory. Third, a solution of a generalized eigenvalue problem for fractional integrated process is provided.

The paper is organized as follows. Section 2 presents data generating process. Section 3 describes the model. In section 4 the asymptotic results and the solution of the generalized eigenvalue problem are presented. Section 5 concludes.

2 Data generating process

Definition 2.1 *Given $p \in \mathbf{N}$, a p-variate time series $\{Y_t\}$ is called fractionally integrated with differencing parameter d ($Y_t \sim I(d)$) , if*

$$Y_t = \sum_{j=0}^{\infty} c_j \epsilon_{t-j} \qquad \text{with } c_j = \frac{\Gamma(j+d)}{\Gamma(j+1)\Gamma(d)}, \tag{1}$$

where $\{\epsilon_t\}_{t>0}$ is an i.i.d. p-variate vector sequence with zero mean.
d is the fractional degree of integration of the process.

The data generating process Y_t is assumed to be a fractional integrated process of order d satisfying the equation (1). Moreover, we assume that $Y_0 = 0$.
We want now to point out the standard distinguishing between the different cases of fractional integrated processes of order d, Y_t, as d varies. We remind the reader to Robinson, (2003).

1. If $d < 1/2$, then Y_t is stationary.

2. If $1/2 < d < 1$, then Y_t is nonstationary and nonpersistent.

3. If $d > 1$, then Y_t is nonstationary and persistent.

Moreover, if $1/2 \leq d < 1$, then Y_t is non explosive, and it is explosive otherwise.

- **Assumption I**

  There exists a p-squared matrix of lag polynomials in the lag operator L such that
  $$\epsilon_t = \sum_{j=0}^{\infty} C_j v_{t-j} =: C(L)v_t, \quad t = 1, \dots, n, \tag{2}$$
  where v_t is a p-variate stationary white noise process.

Now we state some hypotheses on $C(L)$.

- **Assumption II**

  The process ϵ_t can be written as in (2), where v_t are i.i.d. zero-mean p-variate gaussian variables with variance equals to the identity matrix of order p, I_p, and there exist $C_1(L)$ and $C_2(L)$ p-squared matrices of lag polynomials in the lag operator L such that all the roots of $\det C_1(L)$ are outside the complex unit circle and $C(L) = C_1(L)^{-1}C_2(L)$.

The lag polynomial $C(L) - C(1)$ attains value zero at $L = 1$ with fractional algebraic multiplicity equals to d. Thus, there exists a lag fractional polynomial

$$D(L) = \sum_{k=0}^{\infty} D_k L^{\zeta_k}, \qquad D_k,\, \zeta_k \in \mathbf{R},\ \forall\, k = 1, \ldots, +\infty,$$

such that $C(L) - C(1) = (1 - L)^d D(L)$ and ζ_k is increasing.
Therefore, we can write

$$\epsilon_t = C(L)v_t = C(1)v_t + [C(L) - C(1)]v_t = C(1)v_t + D(L)(1 - L)^d v_t. \quad (3)$$

Let us define $w_t := D(L)v_t$. Then, substituting w_t into (3), we get

$$\epsilon_t = C(1)v_t + (1 - L)^d w_t. \quad (4)$$

(4) implies that, given $Y_t \sim I(d)$, we can write recursively

$$\Delta^{d-1}Y_t = \Delta^{d-1}Y_{t-1} + \epsilon_t = \Delta^{d-1}Y_0 + (1 - L)w_t - w_0 + C(1)\sum_{j=1}^{t} v_j. \quad (5)$$

If $\text{rank}(C(1)) = p - r < p$, then the process $\Delta^{d-1}Y_t$ is cointegrated with r linear independent cointegrating vectors $\gamma_1, \ldots, \gamma_r$. Starting by (5), we can show that Y_t is cointegrated with r linear independent cointegrating vectors, that are linear combinations of $\gamma_1, \ldots, \gamma_r$. For sake of simplicity and without losing of generality, we will denote them as $\gamma_1, \ldots, \gamma_r$, as well.

- **Assumption III**

  Let us consider R_r the matrix of the eigenvectors of $C(1)C(1)^T$ corresponding to the r zero eigenvalues. Then the matrix $R_r^T D(1)D(1)^T R_r$ is nonsingular.

Remark 2.1 *Assumption III implies that Y_t cannot be integrated of order $\bar{d}$, with $\bar{d} > d$. In fact, if there exists $\bar{d} > d$ such that $Y_t \sim I(\bar{d})$, then the lag polynomial $D(L)$ admits a unit root with algebraic multiplicity $\bar{d} - d$, and so $D(1)$ is singular. Therefore $R_r^T D(1)D(1)^T R_r$ is singular, and Assumption II does not hold.*

3 The model

The aim of this section is to construct a couple of random matrices, in order to address the solution of the generalized eigenvalue problem.

The random matrices are assumed to be dependent on an integer number $m \geq p$. Let us fix $k = 1, \dots, m$. We define

$$A_m := \sum_{k=1}^{m} a_{n,k} a_{n,k}^T; \tag{6}$$

$$B_m := \sum_{k=1}^{m} b_{n,k} b_{n,k}^T, \tag{7}$$

where

$$a_{n,k} := \frac{M_n^{nonst}/\sqrt{n}}{\sqrt{\int\int F_k(x)F_k(y)\min\{x,y\}\mathrm{d}x\mathrm{d}y}}; \tag{8}$$

$$b_{n,k} := \frac{\sqrt{n}M_n^{st}}{\sqrt{\int F_k(x)^2\mathrm{d}x}}. \tag{9}$$

The terms M_n^{nonst} and M_n^{st} take into account the nonstationary and stationary part of the process, in the sense of the α-th differences of Y_t, α real number. Therefore, we need to consider the distinction pointed out in the data generating process section. Moreover, we recall that

$$Y_t \sim I(d) \Rightarrow \Delta^\alpha Y_t \sim I(d-\alpha). \tag{10}$$

We formalize the presence of two cases.

Case 3.1 *If $d < 1/2$, then $\Delta^\alpha Y_t$ is a nonstationary process for $\alpha \in (-\infty, d - 1/2) \subset (-\infty, 0)$, and it is stationary otherwise.*

Case 3.2 *If $d > 1/2$, then $\Delta^\alpha Y_t$ is a stationary process for $\alpha \in (d-1/2, +\infty) \subset (0, +\infty)$, and it is nonstationary otherwise.*

Then

$$M_n^{nonst} = \frac{1}{n}\sum_{t=1}^{n} F_k(t/n)\Delta^{d-1}Y_t + \int_{-\infty}^{d-1/2} \Big[\phi_1(n,\alpha)\sum_{t=1}^{n} G_{k,\alpha}(t/n)\Delta^\alpha Y_t\Big]\mathrm{d}\alpha; \tag{11}$$

$$M_n^{st} = \frac{1}{n}\sum_{t=1}^{n} F_k(t/n)\Delta^d Y_t + \int_{d-1/2}^{+\infty} \Big[\phi_2(n,\alpha)\sum_{t=1}^{n} H_{k,\alpha}(t/n)\Delta^{\alpha} Y_t\Big] \mathrm{d}\alpha, \qquad (12)$$

with opportunely chosen $\phi_1, \phi_2, F_k, G_{k,\alpha}, H_{k,\alpha}$, we are going to speak about. We need three functional classes. For sake of clarity, we use at this aim two definitions.

Definition 3.1 *Let us fix $m \in \mathbf{N}$, $k = 1, \dots m$.*

(i) There exists a function $\theta_1 : (-\infty, d-1/2) \to \mathbf{R}$ and $\phi_1 : \mathbf{N} \times (-\infty, d-1/2) \to \mathbf{R}$ such that

$$\alpha \mapsto \theta_1(\alpha), \qquad \theta_1 \in L^1(-\infty, d-1/2)$$

and

$$\Big|\sqrt{n}\phi_1(n,\alpha)\sum_{t=1}^{n} t^{\alpha} G_{k,\alpha}(t/n)\Big| \le \theta_1(\alpha), \qquad \forall\, \alpha \in (-\infty, d-1/2).$$

(ii) For each $\alpha \in (-\infty, d-1/2)$, it results

$$\lim_{n\to+\infty} \sqrt{n}\phi_1(n,\alpha)\sum_{t=1}^{n} G_{k,\alpha}(t/n) = 0; \qquad (13)$$

(iii) There exists a function $\theta_2 : (d-1/2, +\infty) \to \mathbf{R}$ and $\phi_2 : \mathbf{N} \times (-\infty, d-1/2) \to \mathbf{R}$ such that

$$\alpha \mapsto \theta_2(\alpha), \qquad \theta_2 \in L^1(d-1/2, +\infty)$$

and

$$\Big|n\phi_2(n,\alpha)\sum_{t=1}^{n} H_{k,\alpha}(t/n)\Big| \le \theta_2(\alpha), \qquad \forall\, \alpha \in (d-1/2, +\infty).$$

(iv) For each $\alpha \in (d-1/2, +\infty)$, it results

$$\lim_{n\to+\infty} n\phi_2(n,\alpha)\sum_{t=1}^{n} t^{\alpha} H_{k,\alpha}(t/n) = 0; \qquad (14)$$

The functional classes $\mathcal{G}_{m,\alpha}$ and $\mathcal{H}_{m,\alpha}$ are

$$\mathcal{G}_{m,\alpha} := \Big\{G_{k,\alpha} : [0,1] \to \mathbf{R} \,|\, (i),\ (ii) \text{ hold}\Big\}. \qquad (15)$$

$$\mathcal{H}_{m,\alpha} := \Big\{H_{k,\alpha} : [0,1] \to \mathbf{R},\ |\, (iii),\ (iv) \text{ hold}\Big\}. \qquad (16)$$

Definition 3.2 *Consider the following conditions:*

$$\frac{1}{\sqrt{n}}\sum_{t=1}^{n} F_k(t/n) = o(1); \tag{17}$$

$$\frac{1}{n\sqrt{n}}\sum_{t=1}^{n} tF_k(t/n) = o(1); \tag{18}$$

$$\int\int F_i(x)F_j(y)\min\{x,y\}\mathrm{d}x\mathrm{d}y = 0, \qquad i \neq j; \tag{19}$$

$$\int F_i(x)\int_0^x F_j(y)\mathrm{d}x\mathrm{d}y = 0, \qquad i \neq j; \tag{20}$$

$$\int F_i(x)F_j(x)\mathrm{d}x = 0, \qquad i \neq j. \tag{21}$$

The functional class $\mathcal{F}_m$ is

$$\mathcal{F}_m := \Big\{F_k : [0,1] \to \mathbf{R},\, F_k \in C^1[0,1] \,|\, (17)\ -\ (21) \text{ hold}\Big\}. \tag{22}$$

Remark 3.1 *Bierens (1997) shows that the functional class $\mathcal{F}_m$ is not empty. He pointed out that, if we define*

$$\bar{F}_k : \mathbf{R} \to \mathbf{R}$$

such that

$$\bar{F}_k(x) = cos(2k\pi x), \tag{23}$$

and taking the restriction

$$F_k := \bar{F}_k|_{[0,1]},$$

then $F_k \in \mathcal{F}_m$.

Remark 3.2 *$\mathcal{G}_{m,\alpha}$ and $\mathcal{H}_{m,\alpha}$ are not empty. There exist infinite suitable functions ϕ_1, ϕ_2, θ_1, θ_2 such that $G_{k,\alpha} \in \mathcal{G}_{m,\alpha}$ and $H_{k,\alpha} \in \mathcal{H}_{m,\alpha}$.*

4 Asymptotic results

In this section the generalized eigenvalue problem is solved. To this end, we define the following p-variate standard normally distributed random vectors:

$$X_k := \frac{\int F_k(x)W(x)\mathrm{d}x}{\sqrt{\int\int F_k(x)F_k(y)\min\{x,y\}\mathrm{d}x\mathrm{d}y}},$$

$$Y_k := \frac{F_k(1)W(1) - \int f_k(x)W(x)\mathrm{d}x}{\int F_k(x)^2\mathrm{d}x},$$

$$X_k^* := \Big(R_{p-r}^T C(1)C(1)^T R_{p-r}\Big)^{\frac{1}{2}} R_{p-r}^T C(1)X_k \sim N_{p-r}(0, I_{p-r}),$$

$$Y_k^* := \Big(R_{p-r}^T C(1)C(1)^T R_{p-r}\Big)^{\frac{1}{2}} R_{p-r}^T C(1)Y_k,$$

$$Y_k^{**} := (R_r^T D(1)D(1)^T R_r)^{-\frac{1}{2}} R_r^T D(1)Y_k \sim N_r(0, I_r).$$

Furthermore, we construct the matrix $V_{r,m}$ as

$$V_{r,m} := (R_r^T D(1)D(1)^T R_r)^{\frac{1}{2}} V_{r,m}^* (R_r^T D(1)D(1)^T R_r)^{\frac{1}{2}},$$

where

$$V_{r,m}^* = \Big(\sum_{k=1}^m \gamma_k^2 Y_k^{**} Y_k^{**T}\Big) - \Big(\sum_{k=1}^m \gamma_k Y_k^{**} X_k^{*T}\Big)\Big(\sum_{k=1}^m X_k^* X_k^{*T}\Big)^{-1}\Big(\sum_{k=1}^m \gamma_k X_k^* Y_k^{**T}\Big).$$

Theorem 4.1 *Assume that $F_k \in \mathcal{F}_m$, $G_{k,\alpha} \in \mathcal{G}_{m,\alpha}$ and $H_{k,\alpha} \in \mathcal{H}_{m,\alpha}$ and Assumptions I, II and III hold.*

- *Let us consider $\hat\lambda_{1,m} \geq \ldots \geq \hat\lambda_{p,m}$ the ordered solutions of the generalized eigenvalue problem*

$$\det\Big[A_m - \lambda(B_m + n^{-2}A_m^{-1})\Big] = 0, \tag{24}$$

and let us consider $\lambda_{1,m} \geq \ldots \geq \lambda_{p-r,m}$ the ordered solutions of the generalized eigenvalue problem

$$\det\Big[\sum_{k=1}^M X_k^* X_k^{*T} - \lambda \sum_{k=1}^M Y_k^* Y_k^{*T}\Big] = 0, \tag{25}$$

where the X_i^*'s and Y_j^*'s are i.i.d. random variables following a $N_{p-r}(0, I_{p-r})$ distribution.

Then we have the following convergence in distribution

$$(\hat\lambda_{1,m}, \ldots, \hat\lambda_{p,m}) \to (\lambda_{1,m}, \ldots, \lambda_{p-r,m}, 0, \ldots, 0).$$

- *Let us consider* $\lambda^*_{1,m} \geq \ldots \geq \lambda^*_{r,m}$ *the ordered solutions of the generalized eigenvalue problem*

$$\det\Big[V^*_{r,m} - \lambda(R_r^T D(1)D(1)^T R_r)^{-1}\Big] = 0. \quad (26)$$

We have the following convergence in distribution

$$n^2(\hat{\lambda}_{p-r+1,m}, \ldots, \hat{\lambda}_{p,m}) \to (\lambda^{*2}_{1,m}, \ldots, \lambda^{*2}_{r,m}).$$

Proof. Due to Lemmas 1, 2 and 4 (Bierens, 1997), it is sufficient to study the asymptotic behavior of $\sqrt{n}M_n^{nonst}$ and nM_n^{st}.

We have

$$\lim_{n\to+\infty} \sqrt{n}M_n^{nonst} = \lim_{n\to+\infty} \frac{1}{\sqrt{n}} \sum_{t=1}^{n} F_k(t/n)\Delta^{d-1}Y_t +$$

$$+ \lim_{n\to+\infty} \int_{-\infty}^{d-1/2} \Big[\phi_1(n,\alpha)\sum_{t=1}^{n} G_{k,\alpha}(t/n)\Delta^{\alpha}Y_t\Big]\mathrm{d}\alpha =: L_1 + L_2$$

By Bierens (1997), we have to show that $L_2 = 0$.

Since $G_{k,\alpha} \in \mathcal{G}_{m,\alpha}$, then the existence of the function θ_1 (defined in (i), Definition 3.1) guarantees, that the Lebesgue Theorem on the dominate convergence holds. Therefore we can write

$$L_2 = \int_{-\infty}^{d-1/2} \lim_{n\to+\infty} \Big[\phi_1(n,\alpha)\sum_{t=1}^{n} G_{k,\alpha}(t/n)\Delta^{\alpha}Y_t\Big]\mathrm{d}\alpha.$$

Moreover, the fractional lag-difference process $\Delta^{\alpha}Y_t$ is well defined, and the condition (ii) of Definition 3.1 assures that $L_2 = 0$, and the first part of the proof is complete.

Now,

$$\lim_{n\to+\infty} M_n^{st} = \lim_{n\to+\infty} \frac{1}{n} \sum_{t=1}^{n} F_k(t/n)\Delta^{d}Y_t +$$

$$+ \lim_{n\to+\infty} \int_{d-1/2}^{+\infty} \Big[\phi_2(n,\alpha)\sum_{t=1}^{n} H_{k,\alpha}(t/n)\Delta^{\alpha}Y_t\Big]\mathrm{d}\alpha =: L_3 + L_4.$$

We have to show that $L_4 = 0$ (Bierens, 1997).

Since $H_{k,\alpha} \in \mathcal{H}_{m,\alpha}$, the existence of the function θ_2 (defined in (ii), Definition

3.1) implies that we are in the hypotheses of the Lebesgue Theorem on the dominate convergence. Thus we have

$$L_4 = \int_{d-1/2}^{+\infty} \lim_{n\to+\infty} \left[\phi_2(n,\alpha)\sum_{t=1}^{n} H_{k,\alpha}(t/n)\Delta^\alpha Y_t\right] \mathrm{d}\alpha.$$

The condition (ii) of Definition 3.1 assures that $L_4 = 0$.
The result is completely proved. ■

5 Conclusions

In this paper a nonparametric cointegration approach for $I(1)$ processes developed by Bierens (1997) is extended to the fractional $I(d)$ process. In order to solve the eigenvalue problem, two random matrices, taking into account the stationary and nonstationary part of the data generating process, are constructed. A general approach, that allows to consider either persistent and antipersistent processes, is proposed. This paper makes three contributions to the growing literature on cointegration analysis. First, a new class of weight functions for the stationary and nonstationary part of the fractional $I(d)$ processes is introduced in a very general framework. Second, the asymptotic results for such random matrices are obtained by using functional analysis theory. Third, a solution of a generalized eigenvalue problem related to the fractional integrated processes is provided.

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