

**A PARAMETRIC CHARACTERIZATION OF  
POLYNOMIAL COFRACTIONALITY IN THE  $VAR_{d,b}(k)$   
MODEL**

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ABSTRACT. We extend the study of the algebraic structure of the  $VAR_{d,b}(k)$  model defined from the fractional lag operator in Johansen (2005) and derive conditions under which its solution is fractional of order  $d$  and displays linear combinations that are fractional of order  $d - b, d - 2b, \dots, d - cb \geq 0$ , for integer  $c$ . We then find the corresponding moving average representation and the cofractional relations.

1. INTRODUCTION

Based on the idea in Granger (1986), Johansen (2005) defines the  $VAR_{d,b}(k)$  model by the following error correction mechanism

$$(1.1) \quad \Delta^d X_t = \alpha\beta' \Delta^{d-b} L_b X_t + \sum_{i=1}^k \Gamma_i \Delta^d L_b^i X_t + \epsilon_t$$

where  $\Delta = 1 - L$ ,  $d \in \mathbb{R}$ ,  $0 < b \leq d$ , and

$$L_b = 1 - (1 - L)^b = L \sum_{n=0}^{\infty} \binom{b}{n+1} (-L)^n,$$

and studies the conditions under which the solution of (1.1) is fractional of order  $d$  and some linear combinations of the process are either fractional of order  $d - b$  or fractional of order  $d - 2b$ . The conditions and representations are analogous to the well known  $I(1)$  and  $I(2)$  cases.

In this paper we extend this approach and derive a complete characterization of the polynomial cofractional relations in the  $VAR_{d,b}(k)$  model; that is, we give conditions on the autoregressive coefficients under which there are linear combinations of the solution of (1.1) that

are fractional of order  $d - b, d - 2b, \dots, d - cb \geq 0$ , for integer  $c$ , and derive the corresponding moving average representation.

## 2. SOME DEFINITIONS AND AN USEFUL TRANSFORMATION

The definitions and the transformation are taken from Johansen (2005). The definition of order of fractionality of a stochastic process is

**Definition 2.1.** *If  $\sum_{i=0}^{\infty} \|C_i\|^2 < \infty$  and  $C(z) = \sum_{i=0}^{\infty} C_i z^i$ ,  $|z| < 1$  can be extended to a continuous function on the boundary  $|z| = 1$ , we call the stationary linear process  $X_t = C(L)\epsilon_t$  fractional of order zero,  $\mathcal{F}(0)$ , if the spectrum at zero  $\omega_X(0) = \frac{1}{2\pi}C(1)\Omega C(1)' \neq 0$ . For such processes we denote with  $\mathcal{F}(0)_+$  the class of asymptotically stationary processes of the form*

$$X_t^+ = \begin{cases} \sum_{i=0}^{t-1} C_i \epsilon_{t-i} & t = 1, 2, \dots \\ 0 & t = 0, -1, \dots \end{cases}.$$

*If  $\Delta_+^d X_t - \mu_t \in \mathcal{F}(0)_+$  for some deterministic function  $\mu_t$  that depends on initial values we say that  $X_t$  is fractional of order  $d$ , and write  $X_t \in \mathcal{F}(d)$ .*

The definitions of cofractionality and polynomial cofractionality are

**Definition 2.2.** *The  $\mathcal{F}(d)$  process  $X_t$  is cofractional if there exists  $\beta$  such that  $\beta' X_t$  is  $\mathcal{F}(d-b)$  with  $0 < b \leq d$ . It is polynomially cofractional if there exists some linear combination of  $X_t$  and its past that is  $\mathcal{F}(e)$  for some  $0 \leq e < d - b$ .*

Resembling the usual concepts in the theory of cointegration, we say that a process is cofractional when it is fractional of a given order and specific linear combinations have a lower order of fractionality; we say that it is polynomially cofractional when it is possible to define a process that has a lower order of fractionality by combing linearly the process at different points in time.

In order to derive the solution of (1.1), one should study the properties of its characteristic polynomial

$$\Pi^*(z) = (1-z)^d I - \alpha \beta' (1-z)^{d-b} (1 - (1-z)^b) - \sum_{i=1}^k \Gamma_i (1-z)^d (1 - (1-z)^b)^i.$$

Johansen (2005) shows that one can instead study the standard characteristic polynomial of the cointegrated VAR model

$$(2.1) \quad \Pi(u) = (1 - u)I - \alpha\beta'u - \sum_{i=1}^k \Gamma_i(1 - u)u^i$$

and relate it to  $\Pi^*(z)$  through the equality  $\Pi(u) = (1 - z)^{b-d}\Pi^*(z)$  and the mapping  $u = 1 - (1 - z)^b$ .

If the roots of  $\det(\Pi(u))$  are either  $u = 1$  or outside of the image of the unit disc under the mapping  $z \mapsto 1 - (1 - z)^b$  and the well known  $I(1)$  condition holds, then the solution of (1.1) is an  $\mathcal{F}(d)$  process that displays linear combinations that are fractional of order  $d - b$  and there is no way of lowering further the order of fractionality by linear combinations. When instead the  $I(1)$  condition fails and the  $I(2)$  holds, different  $\mathcal{F}(d - b)$  processes can be linearly combined and become fractional of order  $d - 2b$  and no further decrement is possible. The intuition would then suggest that if the  $I(2)$  condition is broken and an  $I(3)$  condition formulated, (1.1) would generate an  $\mathcal{F}(d)$  process whose linear combinations would be fractional of order  $d - b$ ,  $d - 2b$  and  $d - 3b$  at most, and so on.

This is what we make precise in Lemma 3.1 below; there we provide the condition on the autoregressive coefficients (the  $I(c)$  condition) under which the solution to (1.1) is an  $\mathcal{F}(d)$  process such that the order of fractionality can be taken down to  $d - b$ ,  $d - 2b, \dots$ ,  $d - cb \geq 0$ , for integer  $c$ , by taking linear combinations. In Lemma 3.2 we give a characterization of the cofractional and polynomial cofractional relations displayed by that process.

### 3. THE ALGEBRAIC STRUCTURE OF THE $VAR_{d,b}(k)$ MODEL

We want to study processes that have either unit or stable roots; that is we require that

$$\det(\Pi(u)) = (u - 1)^m g(u), \quad g(1) \neq 0$$

with  $m > 0$  and that the roots of  $g(u)$  are outside of the image of the unit disc under the mapping  $z \mapsto 1 - (1 - z)^b$  (see Johansen, 2005, for the derivation of the result).

From  $\det(\Pi(1)) = 0$  it follows that  $\Pi(1)$  has reduced rank and thus that it can be written as the product of two non square matrices of

full column rank; that is  $-\delta_1 \gamma'_1 = \Pi(1)$ . Moreover, it follows that the adjoint of  $\Pi(u)$  is

$$\text{adj}(\Pi(u)) = (u - 1)^a H(u)$$

with  $a \geq 0$  and  $H(1) \neq 0$ .

The reason is the following: if  $\text{rank}(\Pi(1)) < p - 1$  any  $p - 1 \times p - 1$  submatrix of  $\Pi(1)$  is singular and thus any minor is zero; if instead  $\text{rank}(\Pi(1)) = p - 1$ , there is at least one  $p - 1 \times p - 1$  non singular submatrix of  $\Pi(1)$  and thus a non zero minor. Since the adjoint is constructed from the minors of  $\Pi(u)$ , it follows that  $u - 1$  can be factored out to some power  $a > 0$  when  $\text{rank}(\Pi(1)) < p - 1$  while  $a = 0$  when  $\text{rank}(\Pi(1)) = p - 1$ .

The inverse is then equal to

$$\Pi(u)^{-1} = \frac{\text{adj}(\Pi(u))}{\det(\Pi(u))} = \frac{H(u)}{(u - 1)^{m-a} g(u)}, \quad u \neq \{u : \det(\Pi(u)) = 0\},$$

where  $H(1) \neq 0$ ,  $g(1) \neq 0$  and  $c = m - a > 0$  is the order of the pole of the inverse function at the unit root. Thus the Laurent expansion around this singularity is<sup>1</sup>

$$\Pi(u)^{-1} = \frac{C_c}{(1 - u)^c} + \frac{C_{c-1}}{(1 - u)^{c-1}} + \cdots + \frac{C_1}{1 - u} + C(u), \quad u \in D(1, \delta) \setminus \{1\}$$

where  $D(1, \delta)$  is the disc centered in  $u = 1$  with radius  $\delta$ ,  $C(u)$  is well defined at  $u = 1$ , and the coefficients are

$$C_{c-v} = \frac{(-1)^{c-v}}{v!} \left( \frac{d}{dz} \right)^v \left( \frac{H(u)}{g(u)} \right) \Big|_{u=1}.$$

Franchi (2006) shows the equivalence between  $m - a = 1$  and  $m - a = 2$  and the well known  $I(1)$  and  $I(2)$  rank conditions in Johansen (1996). The statement about the rank of a matrix which is needed to determine the order of the pole is here replaced with one on the difference of two numbers. It is this simplification that makes the derivation of the general result in Theorem 3.3 possible; in fact the relation between  $\Pi(u)$  and  $\Pi(u)^{-1}$ , the sequence of rank conditions, and the cofractional vectors will appear from the relation  $\Pi(u)\text{adj}(\Pi(u)) = \text{adj}(\Pi(u))\Pi(u) = \det(\Pi(u))I$ , which is now written as

$$(3.1) \quad \Pi(u)H(u) = H(u)\Pi(u) = (u - 1)^c g(u)I.$$

<sup>1</sup>For notational convenience we write  $\Pi$ ,  $H$  and  $g$  instead of  $\Pi(1)$ ,  $H(1)$  and  $g(1)$ ; similarly, we also drop the reference to one in the derivatives, that is we let  $\Pi^{(n)} = \frac{d^n}{dz^n} \Pi(u) \Big|_{u=1}$  and  $H^{(n)} = \frac{d^n}{dz^n} H(u) \Big|_{u=1}$ . When convenient we write  $\dot{\Pi}$  and  $\ddot{\Pi}$  for first and second derivatives.

It is clear that  $v$  is a cofractional vector if and only if it is such that  $v'C_c = 0$ , where  $C_c = \frac{(-1)^c}{g}H$  is the leading coefficient in the principal part of  $\Pi(u)^{-1}$ ; (3.6) in Lemma 3.2 implies

$$C_c = (-1)^c \gamma_{1\perp} \cdots \gamma_{c\perp} A_c^{-1} \delta'_{c\perp} \cdots \delta'_{1\perp},$$

where  $A_c$  is the full rank matrix which provides the Johansen's rank type of condition for integer  $c$  and  $\delta_s$  and  $\gamma_s$  are defined by the sequence of reduced rank matrices in Lemma 3.1. Then the cofractional vectors are given by  $\beta_s = \bar{\gamma}_{1\perp} \cdots \bar{\gamma}_{s-1\perp} \gamma_s$ ,  $s = 1, \dots, c$ .

Since  $(u-1)^c g(u)I$  needs to be differentiated at least  $c$  times to be different from zero at  $u = 1$ , the derivative of order  $n$  of (3.1) at  $u = 1$  immediately provides the following relations

$$(3.2) \quad -\delta_1 \gamma'_1 H^{(n)} + \sum_{v=1}^n \binom{n}{v} \Pi^{(v)} H^{(n-v)} = \begin{cases} 0 & \text{if } n = 1, \dots, c-1 \\ c!gI & \text{if } n = c. \end{cases}$$

The manipulation of these expressions yields the cofractional and polynomial cofractional properties of the process that will be presented in Theorem 4.1. The algebraic results that lead to these are collected in Lemma 3.1, Lemma 3.2 and Theorem 3.3 below.

**Lemma 3.1.** *Let  $c = m - a$ ; then for  $s = 1, \dots, c-1$  the square matrix*

$$(3.3) \quad A_s = \delta'_{s\perp} \cdots \delta'_{1\perp} \theta_1^s \gamma_{1\perp} \cdots \gamma_{s\perp}$$

with

$$(3.4) \quad \theta_v^1 = \Pi^{(v)} \text{ and } \theta_v^s = \theta_1^{s-1} \sum_{j=1}^{s-1} \bar{\beta}_j \bar{\alpha}'_j \theta_v^j + \frac{\theta_{v+1}^{s-1}}{v+1} \text{ for } s \neq 1,$$

where  $\alpha_s = \bar{\delta}_{1\perp} \cdots \bar{\delta}_{s-1\perp} \delta_s$  and  $\beta_s = \bar{\gamma}_{1\perp} \cdots \bar{\gamma}_{s-1\perp} \gamma_s$ , has reduced rank. It is then written as the product of two non square matrices of full column rank

$$-\delta_{s+1} \gamma'_{s+1} = A_s.$$

For  $s = c$ ,

$$(3.5) \quad A_c = \delta'_{c\perp} \cdots \delta'_{1\perp} \theta_1^c \gamma_{1\perp} \cdots \gamma_{c\perp}$$

is full rank.

PROOF. See Appendix. ■

Note that the result specializes for  $c = 1$  into

$$A_1 = \delta'_{1\perp} \dot{\Pi} \gamma_{1\perp}$$

being full rank and for  $c = 2$  into  $\det(A_1) = 0$ ,  $-\delta_2\gamma'_2 = A_1$  and

$$A_2 = \delta'_{2\perp}\delta'_{1\perp}\{\dot{\Pi}\bar{\beta}_1\bar{\alpha}'_1\dot{\Pi} + \frac{\ddot{\Pi}}{2}\}\gamma_{1\perp}\gamma_{2\perp}$$

being full rank, which are the well known  $I(1)$  and  $I(2)$  conditions. In general, as long as  $s < c$ ,  $A_s$  in (3.3) has reduced rank and thus it is written as the product of two non square matrices  $\delta_{s+1}$  and  $\gamma_{s+1}$ .  $A_{s+1}$  is then constructed from the orthogonal complements of  $\delta_1, \dots, \delta_{s+1}$  and  $\gamma_1, \dots, \gamma_{s+1}$  and the matrix  $\theta_1^{s+1}$  which is a complicated function of  $\delta_1, \dots, \delta_s$ ,  $\gamma_1, \dots, \gamma_s$  and  $\theta_1^1, \dots, \theta_1^s$  that can be computed using (3.4). When  $s = c$ ,  $A_c$  becomes full rank, no additional smaller base can be defined and the recursion ends.

The main difficulty in getting these rank properties correctly consists in looking at the right matrices (the  $\theta_1^s$ ) in the right coordinates ( $\delta_{1\perp} \dots \delta_{s\perp}$  and  $\gamma_{1\perp} \dots \gamma_{s\perp}$ ). These rank conditions are very important because they define the sequence of coefficients that characterize cofractionality and polynomial cofractionality.

**Lemma 3.2.** *Let  $c = m - a$ ,  $s = 1, \dots, c$  and  $\beta_s = \bar{\gamma}_{1\perp} \dots \bar{\gamma}_{s-1\perp}\gamma_s$ ; then*

$$(3.6) \quad H = \gamma_{1\perp} \dots \gamma_{c\perp} A_c^{-1} \delta'_{c\perp} \dots \delta'_{1\perp} g,$$

$$(3.7) \quad \beta'_s H = 0,$$

and

$$(3.8) \quad \beta'_s H^{(n)} = \bar{\alpha}'_s \sum_{v=1}^n \binom{n}{v} \theta_v^s H^{(n-v)}, \quad n = 1, \dots, c - s$$

with  $\alpha_s, \delta_s, \gamma_s, \theta_v^s$ , and  $A_s$  as in Lemma 3.1.

PROOF. See Appendix. ■

The cofractional vectors (see (3.7)) are given by  $\beta_s = \bar{\gamma}_{1\perp} \dots \bar{\gamma}_{s-1\perp}\gamma_s$ :  $\beta_1$  is orthogonal to  $\gamma_{1\perp}$ ,  $\beta_2$  lies in that part of  $sp(\gamma_{1\perp})$  which is orthogonal to  $sp(\gamma_{2\perp})$ ,  $\beta_3$  in that part of  $sp(\gamma_{2\perp})$  which is orthogonal to  $sp(\gamma_{3\perp})$ , and so on. At any iteration the space spanned by  $sp(\gamma_{s-1\perp})$  is split into the two orthogonal subspaces given by  $sp(\gamma_s)$  and  $sp(\gamma_{s\perp})$ ; the first is used for constructing  $\beta_s$  and part of the second for  $\beta_{s+1}$ . Smaller and smaller dimensional spaces are met at any iteration until the full rank matrix  $A_c$  is found and no other cofractional vector can be defined.

Next Theorem collects the algebraic results that are needed for the representation of the solution of (1.1).

**Theorem 3.3.** *The coefficients of*

$$\Pi(u)^{-1} = \frac{C_c}{(1-u)^c} + \frac{C_{c-1}}{(1-u)^{c-1}} + \cdots + \frac{C_1}{1-u} + C(u)$$

*satisfy*

$$(3.9) \quad C = (-1)^c \gamma_{1\perp} \cdots \gamma_{c\perp} A_c^{-1} \delta'_{c\perp} \cdots \delta'_{1\perp},$$

$$(3.10) \quad \beta'_s C_c = 0, \beta'_s C_{c-1} \neq 0,$$

*and*

$$(3.11) \quad \beta'_s C_{c-n} = \bar{\alpha}'_s \sum_{v=1}^n \theta_v^s C_{c-n+v}, \quad n = 1, \dots, c-s.$$

*Then*

$$\beta'_s \Pi(u)^{-1}$$

*has a pole of order  $c-1$  at  $u=1$  and*

$$\left\{ \beta'_s - \bar{\alpha}'_s \sum_{v=1}^n \frac{(-1)^v}{v!} \theta_v^s (1-z)^v \right\} \Pi(u)^{-1}$$

*has a pole of order  $s-1$  at  $u=1$ .*

PROOF. See Appendix. ■

Note that the result specializes for  $c=2$  into

$$C_2 = \gamma_{1\perp} \gamma_{2\perp} A_2^{-1} \delta'_{2\perp} \delta'_{1\perp},$$

$$\beta_1 = \gamma_1, \beta_2 = \bar{\gamma}_{1\perp} \gamma_2,$$

so that

$$\beta'_1 C_2 = 0 \text{ and } \beta'_2 C_2 = 0$$

imply that  $\beta'_1 \Pi(u)^{-1}$  and  $\beta'_2 \Pi(u)^{-1}$  have a pole of order one at  $u=1$ . Moreover,

$$\beta'_1 C_1 = \bar{\alpha}'_1 \dot{\Pi} C_2$$

implies that  $\{\beta'_1 + \bar{\alpha}'_1 \dot{\Pi}(1-z)\} \Pi(u)^{-1}$  has no pole left. These are the well known cointegrating relations in the  $I(2)$  model (see Johansen, 1996).

In general, the  $c$  directions in which the pole becomes of order  $c-1$  are given by the  $\beta_1, \beta_2, \dots, \beta_c$  found in Lemma 3.2. When  $c=1$  the only possibility is given by  $\gamma_1 = \beta$  and  $\gamma'_1 \Pi(u)^{-1}$  has no pole at  $u=1$ ; when  $c=2$  both  $\gamma'_1 \Pi(u)^{-1}$  and  $\gamma'_2 \bar{\gamma}'_{1\perp} \Pi(u)^{-1}$  have a pole of order one. The important difference among the  $\beta_s$  can be understood only when we consider polynomial cofractionality. The reason being that

depending on which direction we choose the order of the pole can be reduced differently by taking linear combinations. Think about the well known  $I(2)$  case: in the  $\beta_1 = \gamma_1$  direction we can combine the two  $I(1)$  processes  $\gamma'_1 X_t$  and  $\Delta X_t$  in such a way that their linear combination is stationary but no such way exists in the  $\beta_2 = \bar{\gamma}_{1\perp} \gamma_2$  direction where the only way of reducing  $\gamma'_2 \bar{\gamma}'_{1\perp} X_t$  to stationarity is by differentiation.

This is exactly what happens in the general case (see theorem 3.3): in the  $\beta_1$  direction we eliminate the pole by taking linear combinations, in the  $\beta_2$  direction we have a pole of order one, in the  $\beta_3$  direction of order two, and so on until the  $\beta_c$  direction in which no linear combination can reduce the order of the pole.

#### 4. THE REPRESENTATION THEOREM

**Theorem 4.1** (Representation of cofractional processes). *Let  $c = m - a$  and the roots of  $\det(\Pi^*(z)) = 0$  either at  $z = 1$  or stable; then  $X_t$  has the representation*

$$(4.1) \quad X_t = C_c \Delta_+^{-d} \epsilon_t + C_{c-1} \Delta_+^{-d+b} \epsilon_t + \dots + C_1 \Delta_+^{-d+(c-1)b} \epsilon_t + \Delta_+^{-d+cb} Y_t^+ + \mu_t$$

where the coefficients satisfy (3.10) and (3.11).

The process  $Y_t$  is stationary,  $X_t$  is fractional of order  $d$ ,  $(\beta_1, \beta_2, \dots, \beta_c)' X_t$  is fractional of order  $d - b$  and

$$(4.2) \quad \beta'_s X_t - \bar{\alpha}'_s \sum_{v=1}^n \frac{(-1)^v}{v!} \theta_v^s \Delta_+^{vb} X_t$$

is fractional of order  $d - (c - s + 1)b$ .

PROOF. The result follows from Theorem 3.3; details to be added.

■

From the moving average representation in (4.1) we see that  $X_t$  is composed of  $\mathcal{F}(d - cb)$  up to  $\mathcal{F}(d)$  processes which are generated by cumulating  $\epsilon_t$  and  $Y_t$ . Each of the components is loaded into  $X_t$  through the corresponding  $C$  coefficient and in (3.9) we give the explicit expression of  $C_c$ , which defines the cofractional relations

$$\begin{aligned} \beta'_1 X_t &\sim \mathcal{F}(d - b), \\ \beta'_2 X_t &\sim \mathcal{F}(d - b), \\ &\vdots \\ \beta'_c X_t &\sim \mathcal{F}(d - b). \end{aligned}$$

The other  $C$  coefficients are more complicated and not very interesting in themselves; what is important is to understand which linear combinations of the process have a lower order of fractionality. These are the polynomial cofractional relations described in (4.2) which state that

$$\begin{aligned} \beta'_1 X_t - \bar{\alpha}'_1 \sum_{v=1}^{c-1} \frac{(-1)^v}{v!} \theta_v^1 \Delta_+^{vb} X_t &\sim \mathcal{F}(d - cb), \\ \beta'_2 X_t - \bar{\alpha}'_2 \sum_{v=1}^{c-2} \frac{(-1)^v}{v!} \theta_v^2 \Delta_+^{vb} X_t &\sim \mathcal{F}(d - (c-1)b), \\ &\vdots \\ \beta'_{c-1} X_t + \bar{\alpha}'_{c-1} \theta_1^{c-1} \Delta_+^b X_t &\sim \mathcal{F}(d - 2b). \end{aligned}$$

So we see that in the  $\beta_1$  direction we can combine the process in such a way that we go from  $\mathcal{F}(d)$  to  $\mathcal{F}(d - cb)$ , in the  $\beta_2$  direction from  $\mathcal{F}(d)$  to  $\mathcal{F}(d - (c-1)b)$ , and so on up to the  $\beta_c$  direction in which no polynomial cofractionality is present.

Note that the result specializes for  $c = 2$  into

$$\begin{aligned} X_t &= C_2 \Delta_+^{-d} \epsilon_t + C_1 \Delta_+^{-d+b} \epsilon_t + \Delta_+^{-d+2b} Y_t^+ + \mu_t, \\ (\beta_1, \beta_2)' X_t &\sim \mathcal{F}(d - b), \end{aligned}$$

and

$$\beta'_1 X_t + \bar{\alpha}'_1 \dot{\Pi} \Delta_+^b X_t \sim \mathcal{F}(d - 2b)$$

which is found in Theorem 10 in Johansen (2005).

## 5. CONCLUSION

We have extended the study of the vector autoregressive model that generate fractional, cofractional, and polynomial cofractional processes. The model appears extremely flexible, since it is able to generate processes that possess an very rich cofractional structure; this model could then be used to study economic and financial series in a model based framework.

## APPENDIX

PROOF OF LEMMA 3.1 AND LEMMA 3.2. The proofs of the two Lemmas are done jointly; (3.3), (3.4) and (3.8) are proved by induction on  $s$ , so we begin by proving these results for  $s = 1$ .

For convenience we rewrite (3.2) here

$$(5.1) \quad -\delta_1 \gamma_1' H^{(n)} + \sum_{v=1}^n \binom{n}{v} \Pi^{(v)} H^{(n-v)} = 0, \quad n = 1, \dots, c-1.$$

Pre-multiply it by  $\bar{\delta}'_1$ , let  $\beta_1 = \gamma_1$ ,  $\alpha_1 = \delta_1$ ,  $n_1 = n$  and  $\theta_v^1 = \Pi^{(v)}$ ; then

$$(5.2) \quad \beta_1' H^{(n_1)} = \bar{\alpha}'_1 \sum_{v=1}^{n_1} \binom{n_1}{v} \theta_v^1 H^{(n_1-v)}$$

proves (3.4) and (3.8) for  $s = 1$ .

Now let  $n = 1$  in (5.1), substitute  $H = \gamma_{1\perp} \phi_1 \delta'_{1\perp}$  and  $\theta_1^1 = \dot{\Pi}$  to have

$$-\delta_1 \gamma_1' \dot{H} + \theta_1^1 \gamma_{1\perp} \phi_1 \delta'_{1\perp} = 0;$$

pre and post-multiplication by  $\delta'_{1\perp}$  and  $\bar{\delta}'_{1\perp}$  gives

$$\delta'_{1\perp} \theta_1^1 \gamma_{1\perp} \phi_1 = 0$$

and since  $\det(\delta'_{1\perp} \theta_1^1 \gamma_{1\perp}) \neq 0$  contradicts  $\phi_1 \neq 0$ ,  $A_1 = \delta'_{1\perp} \theta_1^1 \gamma_{1\perp}$  must be of reduced rank. This completes the proof of (3.3), (3.4) and (3.8) for  $s = 1$ .

From  $A_1 \phi_1 = \phi_1 A_1 = 0$  (see (3.1)) and

$$(5.3) \quad -\delta_2 \gamma_2' = A_1$$

for two non square matrices  $\delta_2$  and  $\gamma_2$  of full column rank, it follows that  $\phi_1 = \gamma_{2\perp} \phi_2 \delta'_{2\perp}$  for some  $\phi_2 \neq 0$  and thus that

$$H = \gamma_{1\perp} \gamma_{2\perp} \phi_2 \delta'_{2\perp} \delta'_{1\perp}.$$

To see how the proof works in general we now discuss the case  $s = 2 < c$ ; the second derivative of (3.1) at  $u = 1$  is

$$\Pi \ddot{H} + 2\dot{\Pi} \dot{H} + \ddot{\Pi} H = 0$$

that is (see (5.10) below)

$$-\delta_1 \gamma_1' \ddot{H} + 2\theta_1^1 \dot{H} + \theta_2^1 H = 0.$$

Pre-multiplying it by  $\delta'_{1\perp}$  we have that (see (5.11) below)

$$(5.4) \quad 2\delta'_{1\perp} \theta_1^1 \dot{H} + \delta'_{1\perp} \theta_2^1 H = 0$$

and by the identity  $I = \gamma_{1\perp}\bar{\gamma}'_{1\perp} + \bar{\gamma}_1\gamma'_1$  and  $\beta_1 = \gamma_1$  that

$$\delta'_{1\perp}\theta_1^1\dot{H} = \delta'_{1\perp}\theta_1^1\gamma_{1\perp}\bar{\gamma}'_{1\perp}\dot{H} + \delta'_{1\perp}\theta_1^1\bar{\beta}_1\beta'_1\dot{H} = \underbrace{A_1\bar{\gamma}'_{1\perp}\dot{H}}_{\text{part A}} + \underbrace{\delta'_{1\perp}\theta_1^1\bar{\beta}_1\beta'_1\dot{H}}_{\text{part B}}.$$

By (5.3) *part A* is equal to

$$A_1\bar{\gamma}'_{1\perp}\dot{H} = -\delta_2\gamma'_2\bar{\gamma}'_{1\perp}\dot{H},$$

and since

$$\beta'_1\dot{H} = \bar{\alpha}'_1\theta_1^1H,$$

by (5.2), *part B* becomes

$$\delta'_{1\perp}\theta_1^1\bar{\beta}_1\beta'_1\dot{H} = \delta'_{1\perp}\theta_1^1\bar{\beta}_1\bar{\alpha}'_1\theta_1^1H.$$

Letting  $\beta_2 = \bar{\gamma}_{1\perp}\gamma_2$ , we then have that

$$\delta'_{1\perp}\theta_1^1\dot{H} = -\delta_2\beta'_2\dot{H} + \delta'_{1\perp}\theta_1^1\bar{\beta}_1\bar{\alpha}'_1\theta_1^1H$$

and thus that (5.4) becomes (see (5.12) below)

$$(5.5) \quad -\delta_2\beta'_2\dot{H} + \delta'_{1\perp}\theta_1^2H = 0$$

having set

$$\theta_1^2 = \theta_1^1\bar{\beta}_1\bar{\alpha}'_1\theta_1^1 + \frac{\theta_1^1}{2},$$

which proves (3.4) for  $s = 2$ .

Now pre-multiply (5.5) by  $\bar{\delta}'_2$  and let  $\alpha_2 = \bar{\delta}_{1\perp}\delta_2$  to have (see (5.13) below)

$$\beta'_2\dot{H} = \bar{\alpha}'_2\theta_1^2H,$$

which proves (3.8) for  $s = 2$ .

Now pre and post-multiply (5.5) by  $\delta'_{2\perp}$  and  $\bar{\eta}_2$  and use  $H = \zeta_2\phi_2\eta'_2$  to get

$$\eta'_2\theta_1^2\zeta_2\phi_2 = 0.$$

Since  $|\eta'_2\theta_1^2\zeta_2| \neq 0$  contradicts  $\phi_2 \neq 0$ ,  $\eta'_2\theta_1^2\zeta_2$  must have reduced rank and thus it can be written as

$$-\delta_3\gamma'_3 = \eta'_2\theta_1^2\zeta_2.$$

Then  $\delta_3\gamma'_3\phi_2 = \phi_2\delta_3\gamma'_3 = 0$  follow from the two versions of the identity (3.1) and imply

$$\gamma_{3\perp}\phi_3\delta'_{3\perp} = \phi_2$$

for some  $\phi_3 \neq 0$ . This completes the proof of the statement for  $s = 2$ .

Now we show that if the statement holds for  $s = 1, \dots, k$  then it holds for  $s = k + 1$  for any  $k$ . Let  $\beta_k = \bar{\zeta}_{k-1}\gamma_k$ ,  $\zeta_k = \zeta_{k-1}\gamma_{k\perp}$ ,

$\bar{\alpha}_k = \eta_{k-1}\bar{\delta}_k$ ,  $\eta_k = \eta_{k-1}\delta_{k\perp}$ ,  $n_k = n + 1 - k$  and write (3.8) - (3.4) for  $s = k$ ; that is

$$(5.6) \quad \beta'_k H^{(n_k)} = \bar{\alpha}'_k \sum_{v=1}^{n_k} \binom{n_k}{v} \theta_v^k H^{(n_k-v)},$$

$$(5.7) \quad -\delta_{k+1}\gamma'_{k+1} = \eta'_k \theta_1^k \zeta_k,$$

$$(5.8) \quad \gamma_{k+1\perp} \phi_{k+1} \delta'_{k+1\perp} = \phi_k,$$

and

$$(5.9) \quad \theta_v^k = \theta_1^{k-1} \sum_{j=1}^{k-1} \bar{\beta}_j \bar{\alpha}'_j \theta_v^j + \frac{\theta_{v+1}^{k-1}}{v+1}.$$

Substituting  $\bar{\alpha}_k = \eta_{k-1}\bar{\delta}_k$  into (5.6) we see that

$$(5.10) \quad -\delta_k \beta'_k H^{(n_k)} + \eta'_{k-1} \sum_{v=1}^{n_k} \binom{n_k}{v} \theta_v^k H^{(n_k-v)} = 0$$

and by pre-multiplying it by  $\delta'_{k\perp}$ , letting  $\eta_k = \eta_{k-1}\delta_{k\perp}$ , changing index in the summation, and using  $n_{k+1} = n_k - 1$  that

$$(5.11) \quad \binom{n_k}{1} \eta'_k \theta_1^k H^{(n_{k+1})} + \eta'_k \sum_{v=1}^{n_{k+1}} \binom{n_k}{v+1} \theta_{v+1}^k H^{(n_{k+1}-v)} = 0.$$

By the identity  $I = \zeta_k \bar{\zeta}'_k + \sum_{j=1}^k \bar{\beta}_j \beta'_j$  we write

$$\eta'_k \theta_1^k H^{(n_{k+1})} = \eta'_k \theta_1^k \zeta_k \bar{\zeta}'_k H^{(n_{k+1})} + \eta'_k \theta_1^k \sum_{j=1}^k \bar{\beta}_j \beta'_j H^{(n_{k+1})};$$

by (5.7) we have that

$$\eta'_k \theta_1^k \zeta_k \bar{\zeta}'_k H^{(n_{k+1})} = -\delta_{k+1} \gamma'_{k+1} \bar{\zeta}'_k H^{(n_{k+1})}$$

and by (3.8) for  $j = 1, \dots, k$  that

$$\beta'_j H^{(n_{k+1})} = \bar{\alpha}'_j \sum_{v=1}^{n_{k+1}} \binom{n_{k+1}}{v} \theta_v^j H^{(n_{k+1}-v)}$$

which means that

$$\eta'_k \theta_1^k \sum_{j=1}^k \bar{\beta}_j \beta'_j H^{(n_{k+1})} = \eta'_k \theta_1^k \sum_{j=1}^k \bar{\beta}_j \bar{\alpha}'_j \sum_{v=1}^{n_{k+1}} \binom{n_{k+1}}{v} \theta_v^j H^{(n_{k+1}-v)}.$$

Rearranging terms and setting  $\beta_{k+1} = \bar{\zeta}_k \gamma_{k+1}$ , we then have that

$$\eta'_k \theta_1^k H^{(n_{k+1})} = -\delta_{k+1} \beta'_{k+1} H^{(n_{k+1})} + \eta'_k \sum_{v=1}^{n_{k+1}} \binom{n_{k+1}}{v} \theta_1^k \sum_{j=1}^k \bar{\beta}_j \bar{\alpha}'_j \theta_v^j H^{(n_{k+1}-v)}$$

which implies that (5.11) can be written as

$$(5.12) \quad -\delta_{k+1}\beta'_{k+1}H^{(n_{k+1})} + \eta'_k \sum_{v=1}^{n_{k+1}} \binom{n_{k+1}}{v} \theta_v^{k+1} H^{(n_{k+1}-v)} = 0$$

where

$$\theta_v^{k+1} = \theta_1^k \sum_{j=1}^k \bar{\beta}_j \bar{\alpha}'_j \theta_v^j + \frac{\theta_{v+1}^k}{v+1}$$

and hence (3.4) holds for  $s = k + 1$ .

Pre-multiplying (5.12) by  $\bar{\delta}'_{k+1}$  and setting  $\bar{\alpha}_{k+1} = \eta_k \bar{\delta}_{k+1}$ , we see that

$$(5.13) \quad \beta'_{k+1}H^{(n_{k+1})} = \bar{\alpha}'_{k+1} \sum_{v=1}^{n_{k+1}} \binom{n_{k+1}}{v} \theta_v^{k+1} H^{(n_{k+1}-v)}$$

which is (3.8) for  $s = k + 1$ .

To see that also (??) and (??) hold for  $s = k + 1$ , note that the repeated application of (5.8) implies  $H = \zeta_{k+1}\phi_{k+1}\eta'_{k+1}$ ; now let  $n_{k+1} = 1$  in (5.12), pre and post-multiply it by  $\delta'_{k+1\perp}$  and  $\bar{\eta}_{k+1}$  to get

$$\eta'_{k+1}\theta_1^{k+1}\zeta_{k+1}\phi_{k+1} = 0.$$

Since  $|\eta'_{k+1}\theta_1^{k+1}\zeta_{k+1}| \neq 0$  contradicts  $\phi_{k+1} \neq 0$ ,  $\eta'_{k+1}\theta_1^{k+1}\zeta_{k+1}$  must have reduced rank and thus it can be written as

$$-\delta_{k+2}\gamma'_{k+2} = \eta'_{k+1}\theta_1^{k+1}\zeta_{k+1}$$

proving that (??) holds for  $s = k+1$ . Then  $\delta_{k+2}\gamma'_{k+2}\phi_{k+1} = \phi_{k+1}\delta_{k+2}\gamma'_{k+2} = 0$  follow from the two versions of the identity (3.1) and imply

$$\gamma_{k+2\perp}\phi_{k+2}\delta'_{k+2\perp} = \phi_{k+1}$$

for some  $\phi_{k+2} \neq 0$ , which is (??) for  $s = k + 1$ . Then (3.8) - (3.4) hold for  $s = k + 1$  and the induction part of the proof is complete.

To see that (??) is true, note that using the previous recursion the derivative of order  $c$  can be written as

$$-\delta_c\gamma'_c\dot{H} + \eta'_{c-1}\theta_1^c H = gI.$$

Pre-multiplying by  $\delta'_{c\perp}$  and using  $\eta_c = \eta_{c-1}\delta_{c\perp}$ , we have  $\eta'_c\theta_1^c H = g\delta'_{c\perp}$ . Since  $H = \zeta_c\phi_c\eta'_c$  and  $\delta'_{c\perp}\bar{\eta}_c = I$ , post-multiplication by  $\bar{\eta}_c$  turns it into

$$\eta'_c\theta_1^c\zeta_c\phi_c = gI$$

and the proof is complete. ■

PROOF OF THEOREM 3.3. (3.10) and (3.11) follow from Lemma 3.2 (details to be added). The Taylor expansion of  $H(u)$  at  $u = 1$  is written as

$$H(u) = \sum_{v=0}^{c-s} \frac{H^{(v)}}{v!} (u-1)^v + (u-1)^{c-s+1} A(u)$$

and then

$$(5.14) \quad \beta'_s H(u) = \beta'_s \sum_{v=1}^{c-s} \frac{H^{(v)}}{v!} (u-1)^v + (u-1)^{c-s+1} \beta'_s A(u)$$

follows from  $\beta'_s H = 0$ .

Using (3.8) and rearranging terms, we have that

$$\beta'_s \sum_{v=1}^{c-s} \frac{H^{(v)}}{v!} (u-1)^v = \bar{\alpha}'_s \sum_{v=1}^{c-s} \frac{\theta_v^s}{v!} \sum_{k=v}^{c-s} \frac{H^{(k-v)}}{(k-v)!} (u-1)^k$$

and since

$$(u-1)^v H(u) = \sum_{k=v}^{c-s} \frac{H^{(k-v)}}{(k-v)!} (u-1)^k + (u-1)^{c-s+1} B(u)$$

we have that

$$\beta'_s \sum_{v=1}^{c-s} \frac{H^{(v)}}{v!} (u-1)^v = \bar{\alpha}'_s \sum_{v=1}^{c-s} \frac{\theta_v^s}{v!} (u-1)^v H(u) + (u-1)^{c-s+1} C(u).$$

Then (5.14) is rewritten as

$$(5.15) \quad \left\{ \beta'_s - \bar{\alpha}'_s \sum_{v=1}^{c-s} \frac{\theta_v^s}{v!} (u-1)^v \right\} H(u) = (u-1)^{c-s+1} D(u).$$

Dividing both sides of (5.15) by  $(u-1)^d g(u)$  we have that

$$\left\{ \beta'_s - \bar{\alpha}'_s \sum_{v=1}^{c-s} \frac{\theta_v^s}{v!} (u-1)^v \right\} \Pi(u)^{-1} = \frac{D(u)}{(u-1)^{s-1} g(u)}$$

and since  $\frac{D(1)}{g(1)} \neq 0$  the pole at  $u = 1$  has order  $s-1$ . Since the difference operator is defined as  $\Delta = 1 - L$  we use  $(-1)$  to turn  $u-1$  into  $1-u$  and the proof is complete. ■

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