Testing for Short-run and Long-run Asymmetry in ARFIMA models

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Abstract

In this paper, we introduce a new class of models called Threshold ARFIMA (Fractionally Integrated ARMA) models. We test for threshold in the short and long term parameters of ARFIMA models. We apply our framework to the G7 market indexes returns as well as the world market index returns. Long memory was detected in four of the eight series studied in this paper. All the previous series present an asymmetric behaviour in their long term horizon and, except the FTSE index, in their short term behaviour. We show the superiority of our Threshold ARFIMA models over the symmetric ARFIMA via a forecasting study.

1 Introduction

Fractionaly integrated models were largely used to take into account the long memory property of economic and financial times series. Particularly, ARFIMA models were associated with long memory in the representation of the conditional mean of time series data. They were introduced in the econometric litterature by Granger and Joyeux (1980) and Hosking (1981) to take into account the presence of a long memory component in the conditional mean of a time series. A time series is said to exhibit long memory if its autocorrelations decay at a slower rate than the exponential rate related to an ARMA process. They are a generalization of the ARIMA model of Box and Jenkins (1976) caracterized by a differencing parameter taking integar values. In the ARFIMA models, the differencing parameter is allowed to take real values ranging from $-\frac{1}{2}$ to $\frac{1}{2}$. Until now, only symmetric ARFIMA were considered.

Empirically, long memory can be defined in term of the presence of observed correlations. McLeod and Whittle (1978) show that the process y_t exhibit long memory if the sum of the absolute value of its autocorrelation function is non-finite. The class of long memory processes includes every process having an autocovariance function of type $\lambda_k = \Theta(k)k^{2H-2}$ where $\Theta(k)$ is a slowly varying function at infinity[1]. The contribution of this paper is to note that asymmetry is present either in the short memory or in the long memory parameters.

The paper is organized as follows: In section 2, we present the symmetric ARFIMA models and give their propoerties. We introduce, in section 3, the new class of threshold or asymmetric ARFIMA models where we distinguish between short-run and long-run asymmetry. In section 4 we present estimation techniques used in the empirical part of the paper. Section 5 is devoted to a litterature review on threshold models. In section 6, we use a Monte Carlo study to assess the power and the size of the threshold test. In section 7, we fit the best ARFIMA (p,d,q) representation for the stock index returns of the G7 countries as well as the world stock market index returns and apply the threshold test to the series exhibting long memory. Section 8 is devoted to a forecasting study in order to show the superiority of the threshold ARFIMA models over the symmetric ARFIMA. We conclude in the last section.

2 Porperties of ARFIMA process

The ARFIMA(p,d,q) process is defined by Granger(1980), Granger and Joyeux(1980) and hosking (1981) as follows:

$$\Phi(L)(1-L)^d x_t = \theta(l)\varepsilon_t \tag{1}$$

where $E(\varepsilon_t) = 0$, $E(\varepsilon_t^2) = \sigma^2$ and $E(\varepsilon_t \varepsilon_s) = 0$ for $t \neq s$. $\Phi(L)$ and $\theta(L)$ are polynomials defined in the lag operator L, of orders p and q, respectively, and have all their roots outside the unit circle. When $p = q = 0$, the process is reduced to the ARFIMA(0,d,0) process and it can be written as follows:

$$\nabla^d x_t = \varepsilon_t \tag{2}$$

where ∇^d is the following expansion:

$$\nabla^d = (1-L)^d = 1 - dL - \frac{1}{2}d(1-d)L^2 - \frac{1}{6}d(1-d)(2-d)L^3 - ... \tag{3}$$

$$= \sum_{j=0}^{\infty} \pi_j L^j \tag{4}$$

and π^j could be written as :

$$\pi^j = \prod_{0<k\leq j} (\frac{k-d-1}{k}) \text{ and } j = 0, 1, ... \tag{5}$$

Bhardwaj and Swanson (2006) give values for the coefficients in Eq. (5) associated with different lags in the expansion of $(1-L)^d$.

[1]A function is slowly varying function at infinity if $\lim_{t\to\infty}[f(tx)/f(t)]$=x^0

As particular cases, the ARFIMA model reduces to a simple ARMA model when $d = 0$ and to a simple ARIMA model when $d = 1$.

Different modelizations are considered according to the value of d. For $0 < d < 0.5$, the ARFIMA(0,d,0) process is stationary and has a wold decomposition or infinite-order moving average representation given by $x_t = \sum_{i+0}^{\infty} \Psi_i \varepsilon_{t-i}$ where:[2]

$$\Psi_i = \frac{(i+d-1)!}{i!(d-1)!} \tag{6}$$

For $d = 0$, the ARFIMA process reduces to the standard ARMA process and for $-\frac{1}{2} < d < 0$, the ARFIMA(0,d,0) process is invertible and has the infinite autoregressive representation given by $\varepsilon_t = \sum_{i=0}^{\infty} \pi_i x_{t-i}$ where π_k is given by:[3]

$$\pi_i = \frac{(i-d-1)!}{i!(-d-1)!} \tag{7}$$

3 Threshold ARFIMA models

In this section we introduce a new class of ARFIMA models called threshold ARFIMA. Our purpose is to test for the presence of asymmetry in two levels, the short range parameters in a first step and the long range range parameter in a second step.

3.1 short range asymmetry

Introducing a threshold effect in the autoregressive parameters of Eq (1) yields to the following general form:

$$\Phi^+(L)(1-L)^d x_t^+ + \Phi^-(L)(1-L)^d x_t^- = \theta(l)\varepsilon_t \tag{8}$$

where $E(\varepsilon_t) = 0$, $E(\varepsilon_t^2) = \sigma^2$ and $E(\varepsilon_t \varepsilon_s) = 0$ for $t \neq s$. $\Phi^+(L)$ and $\Phi^-(L)$ are polynomials of order p describing the dynamics of the autoregressive part of the ARFIMA model and are such that : $\Phi^+(L) = 1 - \Phi_1^+ L - \Phi_2^+ L^2 - ... - \Phi_p^+ L^p$ and $\Phi^-(L) = 1 - \Phi_1^- L - \Phi_2^- L^2 - ... - \Phi_p^- L^p$, $\theta(L)$ is a polynomial describing the moving average part of the model. These polynomials are defined in the lag operator L, and have all their roots outside the unit circle.
$x_t^+ = x_t I_{\varepsilon_t > \gamma}$ and $x_t^- = x_t I_{\varepsilon_t < \gamma}$. γ is the threshold describing the asymmetric effects of a random shock ε_t.

For the Monte Carlo experiment in section 4, we consider a threshold ARFIMA (1,d,0) model so that only the threshold effect of the autoregressive parameters will be detected. The ARFIMA (1,d,0) model could be written as follow:

[2] As $i \to \infty$, $\Psi_i \to k i^{-d-1}$

[3] As $i \to \infty$, $\pi_i \to i^{-d-1}$

$$
\begin{aligned}
(1-\Phi_1)(1-L)^d &= \theta(l)\varepsilon_t \ if \ \varepsilon_t > 0 \\
(1-\Phi_1)(1-L)^d &= \theta(l)\varepsilon_t \ if \ \varepsilon_t < 0
\end{aligned} \tag{9}
$$

In the previous representation, we have assumed that $\Phi^+(L)$ and $\Phi^-(L)$ are of order 1 and the threshold is null.

3.2 long range asymmetry

Our purpose in this subsection is to test for asymmetry in the long term parameter. We consider the following asymmetric ARFIMA(0,d$^+$,d$^-$,0) model.

$$
\begin{aligned}
(1-L)^{d^+} x_t &= \varepsilon_t \ if \ \varepsilon_t > \gamma \\
(1-L)^{d^-} x_t &= \varepsilon_t \ if \ \varepsilon_t < \gamma
\end{aligned} \tag{10}
$$

where d^+ and d^- are fractional integration parameters in regime 1 and 2, respectively. L, x_t, ε_t and γ are as defined above.

Short range asymmetry was studied on several occasions and in our knowledge, long run asymmetry has never been considered untill now. That's why we have had the idea and the motivation to investigate this area.

4 Estimation of long memory parameter

4.1 GPH estimator

Geweke and Porter-Hudak (1983) prposed a simple method to test for long memory and estimate the fractional differencing parameter. Consider the following model $(1-L)^d x_t = \varepsilon_t$, where ε_t is a stationary linear process with a finite spectral density function $f_\varepsilon(\lambda)$.The spectral density function of $\{X_t\}$ is :

$$
f(\lambda) = (\sigma^2/2\pi)\{4\sin^2(\lambda/2)\}^{-d} f_\varepsilon(\lambda) \tag{11}
$$

and

$$
\ln\{f(\lambda)\} = \ln\{\sigma^2 f_\varepsilon(0)/2\pi\} - d\ln\{4\sin^2(\lambda)\} + \{f_\varepsilon(\lambda)/f_\varepsilon(0)\} \tag{12}
$$

Let $\lambda_{j,T} = 2\pi j/T$, $j = 0, 1, ...T-1$, denote the harmonic ordinates, T is the sample size and $I(\lambda_{j,T})$ denote the periodogram at these ordinates. The sample periodogram $I(\lambda_{j,T})$ is obtained as follow:

$$
I(w_j) = \frac{1}{2\pi T}\left|\sum_{t=1}^{T} y_t e^{-e w_j t}\right|^2 \tag{13}
$$

Evaluation of (8) leads to the following expression:

$$\begin{aligned} \ln\{I(\lambda_{j,T})\} &= \ln\{\sigma^2 f_\varepsilon(0)/2\pi\} - d\ln\{4\sin^2(\lambda_{j,T}/2)\} + \\ &\quad \ln\{f_\varepsilon(\lambda_{j,T})/f_\varepsilon(0) + \ln\{I(\lambda_{j,T})/f(\lambda_{j,T})\} \end{aligned} \tag{14}$$

The GPH estimator is $-d$, it can be etimated by Ordinary Least Square. $n = g(T)$, the number of ordinates, must satisfy a number of conditions in order to have an OLS estimator $\widehat{d}$ asymptotically normal. In fact, $g(T)$ should be such that $\lim_{T\to\infty} g(T) = \infty$ and $\lim_{T\to\infty} g(T)/T = 0$.

The second step in the GPH procedure consists in fitting an ARMA model after filtring the data by replacing the parameter d in (2) by $\widehat{d}_{GPH}$.

4.2 Exact maximum likelihood estimator

Many authors have considered joint estimation of the parameters in the ARFIMA (p,d,q) model under the assumption of normality. The parameter vector $\lambda' = \mu\beta'$ where $\beta^{'} = (\phi_1, ..., \phi_p,\ \theta_1, ..., \theta_q, d, \sigma^2)$ is of dimension (p+q+2). Li and McLeod (1986) showed that, assuming that $\mu = 0$ and $\sigma^2 = 1$, the asymptotic distribution of the MLE estimator is normal with mean zero and covariance matrix I^{-1}.They assert that

$$T^{\frac{1}{2}}(\widehat{\beta} - \beta) \to N[0, I^{-1}]$$

where I is the information function given by:

$$I = \begin{bmatrix} I_{p,q} & J \\ J^{'} & \frac{\pi^2}{6} \end{bmatrix}$$

where $I_{p,q}$ is the usual information matrix of an ARMA process and $J = [\gamma_0^{ud}, ..., \gamma_{p-1}^{ud}, \gamma_0^{vd}, ..., \gamma_{q-1}^{vd}]$

where

$$\gamma_j^{ud} = \sum_{i=0}^{\infty} \tfrac{a_i}{i+j+1}$$
$$\gamma_j^{vd} = \sum_{i=0}^{\infty} \tfrac{b_i}{i+j+1}$$
$$\phi^{-1}(L) = \sum a_i L^i$$
$$\theta^{-1}(L) = \sum b_i L^i$$

Sowell (1992b) provides a recursive procedures allowing for an efficient evaluation of the likelihood function. Unlike the two-step procedures, this method estimates all the parameters of an ARFIMA model in one step. Previously, Li and MCLeod (1986) noted that the small sample behaviour of this type of estimator is poor if the spectrum of the series contains peaks near zero, which is known to exist for positive values of the fractional differencing parameter. They conclude that the maximum likelihood is optimal.

Consider a stationary normally distributed time series z_t generated by the model given in Eq (1) which satisfies a number of assumptions[4]. Let Z_T be a sample of T observations such that $Z_T = [z_1, z_2...z_T]'$ and $Z_T \rightsquigarrow N_T(0, \sum)$, with the probability density function:

$$f(Z_T, \sum) = (2\pi)^{-\frac{T}{2}} \left|\sum\right|^{-\frac{1}{2}} \exp\{-\frac{1}{2} Z_T^{'} \sum^{-1} Z_T\} \tag{15}$$

The log-likelihood function is not globally concave, hense, the results of the search algorithm depend on the choice of starting values. There is no general rule for selecting starting values. Sowell (1992a) presents two procedures for the choice of starting values.

The EML method of Sowell (1992a), in spite of its difficulty and computational requirements is the most efficient method for estimating the parameters of ARFIMA models. The log-likelihood function of an ARFIMA model is written as follow:

$$\log L(d, \phi, \theta, \beta, \sigma_\varepsilon) = -\frac{T}{2} \log(2\pi) - \frac{1}{2} \log|\Sigma| - \frac{1}{2} x' \Sigma^{-1} x \tag{16}$$

[4] see Sowell (1992b) for further details on the assumptions

where x_t is a stationay normally distributed fracrionally integrated time series generated by the equation (1)[5], T is the number of observations and Σ is the covariance matrix of x_t, it is a Toepliz form because of the stationarity of x_t.

$$\Sigma = [\gamma(i-j)] \text{ for } i,j = 1,2,...,T \tag{17}$$

5 Threshold test in ARFIMA models

Two different approaches were considered in threshold models. The first one consists in modelling, explicitly, every regime and considering the previous regimes as observed. This class of models includes the TAR (Threshold AutoRegeressive) model. It was discussed mainly by Tong and Lim (1980) and Tong (1983). The second approach was developed by Hamilton (1989) who has modelled the change in régimes by using a Descrete State Markov process. In this framework, the econometrician could not observe the shift in regime directly but, based on the observed series, he draws a probabilistic inference about the date of occurence of shifts. Application of this approach to postwar US real GNP shows that a periodic shift from a positive growth rate to a negative growth rate is occurent in the the US business cycle.

To test for threshold in the autoregressive parameters and in the long-run parameters of the ARFIMA models, we apply the test of Hansen (1996). In fact, under the null hypothesis of no threshold effect, the threshold model we consider reduces to the symmetric ARFIMA model and it fall in the class of testing problems involving nuisance parameters not identified under the null.

Let recall briefly the hansen's (1996) procedure and test statistics. Let $\{v_t = (y_t, x_t); t = 1, ..., n\}$ be a data set drawn from an underlying probability space and satisfies the regression relationship $y_t = x_t(\gamma)'\theta + \varepsilon_t$ where $x_t(\gamma) = (x_{1t}^{'} h_t(\gamma)')'$ and $\theta = (\theta_1^{'}\theta_2^{'})'$. The purpose is to test the null : $\theta_2 = 0$. A local-by-null reparametrization of the form : $\theta_2 = c/\sqrt{n}$ modifies the testing problem in $H_0 : c = 0$ against $H_1 : c \neq 0$. A heteroscedasticity-robust Lagrange Multiplier test takes the following form:

$$K_n(\gamma) = n\widehat{\theta}(\gamma)'U(U\widehat{V}_n^{*}U)^{-1}U'\widehat{\theta}(\gamma) \tag{18}$$

where U is the selector matrix $U = (0I_p)'$, $\widehat{V}_n^{*}(\gamma) = M_n(\gamma,\gamma)^{-1}\widehat{V}_n(\gamma)M_n(\gamma,\gamma)^{-1}$, $M_n(\gamma_1,\gamma_2) = \frac{1}{n}\sum_{t=1}^{n} x_t(\gamma_1)x_t(\gamma_2)'$ and $\widehat{V}_n(\gamma) = \sum_{t=1}^{n} \widetilde{s}_t(\gamma)\widetilde{s}_t(\gamma)'$ and where $\widetilde{s}_t(\gamma)$ is calculated as $\widetilde{s}_t(\gamma) = x_t(\gamma)\widetilde{\varepsilon}$. $\widetilde{\varepsilon}$ are residuals resulted from the estimation of the model under H_1. The statistic (20) is usuful when γ was known a priori. In our framework γ is unknown and is selected over an interval $\Gamma = [D, U]$. Following Hansen (1996), we use the Davies's (1977,1987) test statistic defined

[5]EML method requires two additional assumptions: (Sowel 1992a)
* d < $\frac{1}{2}$
* The roots of $\Phi(L)$ are simple.

as $SupK_n = \underset{\gamma \in \Gamma}{Sup} K_n(\gamma)$ (21) and the Andrews and Ploberger's (1994) procedures taking the following forms $AveK_n = \int_\Gamma K_n(\gamma)dW(\gamma)$ (22) and $ExpK_n = Ln(\int_\Gamma \exp(\frac{1}{2}(Z(\gamma)dW(\gamma))$ (23). These statistics allow to overcome the problems related to threshold models and mentionned in Hansen (1996) problems.

6 Monte Carlo study

In order to assess the size and the power of our estimates in eq. (21), (22) and (23), we run a simulation experiment with 500 replications in each experience.All the results of this section were obtained with GAUSS.

6.1 Short range case

We report the power of the testing methodology in a simple Monte Carlo simulation study. We use the short-run threshold ARFIMA (p,d,q) model (12) with $p = 1$, $q = 0$ and $d = 0.35$,We fix ϕ_2 at 0.2 and we allow ϕ_1 to take values in $\{-0.8, -0.5, -0.2, 0.1, 0.5, 0.8\}$, ε_t is idd N(0,1) and we consider eleven sample sizes, $T = 200, 300, ...1000, 2000, 3000$. For all sizes, 500 simulated samples were drawn and γ was estimated by GPH and then we fitted and ARMA (1,0) to the filtred series. For every replication, we generate a serie of fractionnally integrated data following an ARFIMA (1,d,0) according to equation (12) using a first seed, say seed 1, to generate the serie of x_t and a second seed, say seed 2 to generate the serie of ε_t. We have assessed the power of the test for three statistics SupLM, AveLM and ExpLM. The number of internal simulation replications was set at $J = 500$. Here, we test the null $H_0 : \phi_1 = \phi_2$ against the alternative $H_1 : \phi_1 \neq \phi_2$.

We found, as reported in table 1, that the power of the test is increasing in T and in $|\phi_1 - \phi_2|$ and presents a tractable power for sample size $T > 1000$.

6.2 Long range case

In this subsection, we assess the size and the power of the test when testing for threshold in the long range parameter. We use the long-range threshold ARFIMA (p, d_1, d_2, q) model (13) with p=q=0. We fix d2 at 0.45 and we allow d1 to vary in the set {0.1, 0.2,0.3,0.4}. ε_t is idd N(0,1) and we consider the same sample sizes as previously. d was estimated by GPH. In every replication, we generate a series of fractionally integrad data following an ARFIMA (0,d1,0.45,0). The test statistics show a good power. As expected, the power increases in the sample size and in |d1-d2|. Table 2 show that the power is tractable for a sample size T>500. The null we test is H_0:$d_1 = d_2$ against the alternative $H_1 : d_1 \neq d_2$. Results for the empirical sizes of the test statistics were collected in table 3 and show that the empirical sizes are much closer to their nominal values when the sample size is bigger.

table 1 :
Finite Sample Power of 5 % size test

	phi(1)=-0.8			phi(2)=-0.5			phi(3)=-0.2		
T	SupLM	ExpLM	AveLM	SupLM	ExpLM	AveLM	SupLM	ExpLM	AveLM
200	0.628	0.744	0.806	0.250	0.370	0.418	0.096	0.140	0.102
300	0.908	0.952	0.950	0.494	0.622	0.640	0.122	0.172	0.216
400	0.974	0.982	0.982	0.734	0.818	0.828	0.212	0.286	0.338
500	0.992	0.992	0.996	0.848	0.924	0.928	0.292	0.384	0.436
600	0.998	0.998	0.998	0.896	0.938	0.958	0.388	0.474	0.498
700	0.996	0.998	0.998	0.966	0.984	0.982	0.442	0.568	0.598
800	0.998	0.998	0.998	0.980	0.992	0.992	0.482	0.606	0.668
900	0.998	0.998	0.998	0.984	0.990	0.988	0.590	0.716	0.734
1000	0.998	0.998	0.998	0.994	0.996	0.996	0.668	0.804	0.800
2000	0.998	0.998	0.998	0.998	0.998	0.998	0.950	0.978	0.982
3000	0.998	0.998	0.998	0.998	0.998	0.998	0.996	0.998	0.996

	phi(4)=0.1			phi(5)=0.4			phi(6)=0.7		
T	SupLM	ExpLM	AveLM	SupLM	ExpLM	AveLM	SupLM	ExpLM	AveLM
200	0.018	0.022	0.040	0.066	0.098	0.124	0.386	0.502	0.614
300	0.044	0.048	0.058	0.132	0.188	0.256	0.674	0.792	0.856
400	0.038	0.040	0.058	0.170	0.234	0.302	0.846	0.908	0.938
500	0.032	0.036	0.062	0.208	0.292	0.348	0.932	0.972	0.984
600	0.044	0.046	0.052	0.270	0.380	0.434	0.966	0.984	0.988
700	0.050	0.058	0.084	0.324	0.450	0.514	0.982	0.988	0.998
800	0.046	0.056	0.080	0.402	0.520	0.612	0.992	0.996	0.996
900	0.072	0.086	0.098	0.454	0.566	0.612	0.992	0.994	0.998
1000	0.078	0.092	0.090	0.518	0.644	0.710	0.998	0.998	0.998
2000	0.080	0.104	0.130	0.906	0.952	0.968	0.998	0.998	0.998
3000	0.132	0.172	0.190	0.984	0.994	0.998	0.998	0.998	0.998

table 2 :
Finite Sample Power of 5 % size test

	d=0.1			d=0.2			d=0.3			d=0.4		
T	SupLM	ExpLM	AveLM	SupLM	ExpLM	AveLM	SupLM	ExpLM	AveLM	SupLM	ExpLM	AveLM
200	0.628	0.744	0.806	0.596	0.706	0.758	0.592	0.700	0.724	0.624	0.714	0.732
300	0.908	0.952	0.950	0.882	0.932	0.930	0.882	0.920	0.922	0.906	0.928	0.928
400	0.974	0.982	0.982	0.964	0.980	0.976	0.962	0.980	0.978	0.970	0.988	0.988
500	0.992	0.992	0.996	0.984	0.990	0.990	0.982	0.986	0.988	0.990	0.992	0.992
600	0.998	0.998	0.998	0.998	0.998	0.998	0.998	0.998	0.998	0.998	0.998	0.998
700	0.996	0.998	0.998	0.996	0.996	0.996	0.996	0.996	0.996	0.998	0.998	0.998
800	0.998	0.998	0.998	0.998	0.998	0.998	0.996	0.996	0.996	0.998	0.998	0.998
900	0.998	0.998	0.998	0.998	0.998	0.998	0.998	0.998	0.998	0.998	0.998	0.998
1000	0.998	0.998	0.998	0.998	0.998	0.998	0.998	0.998	0.998	0.998	0.998	0.998
2000	0.998	0.998	0.998	0.998	0.998	0.998	0.998	0.998	0.998	0.998	0.998	0.998
3000	0.998	0.998	0.998	0.998	0.998	0.998	0.998	0.998	0.998	0.998	0.998	0.998

table 3:
Finite Sample Size of Asymptotic 5 % tests

	d=0.1			d=0.2			d=0.3			d=0.4		
T	SupLM	ExpLM	AveLM	SupLM	ExpLM	AveLM	SupLM	ExpLM	AveLM	SupLM	ExpLM	AveLM
200	0.714	0.714	0.732	0.720	0.718	0.738	0.734	0.722	0.744	0.736	0.728	0.752
300	0.528	0.558	0.632	0.526	0.568	0.628	0.528	0.558	0.632	0.554	0.560	0.664
400	0.364	0.396	0.486	0.364	0.390	0.490	0.368	0.400	0.490	0.370	0.404	0.488
500	0.240	0.266	0.414	0.254	0.270	0.424	0.256	0.292	0.414	0.272	0.306	0.426
600	0.206	0.244	0.404	0.222	0.254	0.402	0.228	0.266	0.430	0.246	0.282	0.444
700	0.169	0.198	0.362	0.184	0.222	0.360	0.204	0.230	0.372	0.204	0.230	0.394
800	0.140	0.172	0.324	0.142	0.160	0.340	0.138	0.168	0.356	0.146	0.178	0.364
900	0.124	0.154	0.272	0.132	0.154	0.282	0.142	0.164	0.294	0.144	0.170	0.30 0
1000	0.100	0.122	0.264	0.100	0.126	0.266	0.098	0.118	0.262	0.114	0.122	0.270
2000	0.056	0.076	0.200	0.058	0.082	0.200	0.058	0.068	0.204	0.058	0.070	0.212
3000	0.036	0.048	0.176	0.034	0.050	0.176	0.038	0.052	0.168	0.040	0.048	0.162

7 Application to stock markets indices

Recent papers have devoted a lot of attention to the behaviour of stock markets returns. Granger and Hyung (2004) found that an occasional break performs better than an I(d) process in terms of in-sample fitting. They showed that there is a causality effect between breaks and long memory in the SP500 series. Shively (2006) estimates a structural bivariate threshold model to distinguish temporary from permanent innovations in positive-return and negative-return regimes. Bhardwaj and Swanson (2006) showed the usefulness and the superiority of the ARFIMA model over a set of non-ARFIMA models to better fit 5 international stock market returns i.e SP500, FTSE, DAX, Nikkei and Hang Sang. All the forecasting horizons they considered showed that the ARFIMA model is preferred to the non ARFIMA models.

In this section, we apply our framework to the G7 stock market indices as well as the MSCI world market index.

7.1 Data analysis

The data examined in this section consists in the indices of the G7 countries as well as the world index MSCI. We consider daily data going from 20 march 1995 till 30 december 2005.

table 4: stationarity test. Philips Perron

data	log(data)	$\Delta \log(data)$
SP500	-2,624310	-54,14071
FTSE	-2,145744	-53,2796
CAC40	1,383267	-52,60648
DAX	0.970924	-50,62815
italie	1,259262	-49,33025
Nikkei	0,533458	-48,11011
Canada	3,177330	-46,65391
MSCI	-2,050900	-43,18792

Estimation by exact maximum likelihood needs the stationarity of data. First of all, we apply the Philips and Perron (1988) Test. Table 4 reports values of test statistic indicating that, in the 5% level, unit root assumption is retained for the log-series and is rejected for returns data. Descriptive statistics reported in table 5 concern differencied log(data).

Table 5: Descriptive statistics

DATA	N. obs	mean	Std. dev	Skewness	Kurtosis
SP500	2000	0.000328	0.11006	-0.104784	6.466351
FTSE	2000	0.000209	0.010859	-0.188160	6.104518
CAC40	2000	0.000340	0.013852	-0.101763	5.811668
DAX	2000	0.000245	0.012023	-0.380386	5.833446
Italy	2000	0.000373	0.013721	-0.084016	6.980937
Nikkei	2000	0.000131	0.012149	-0.230506	5.700646
Canada	2000	0.000625	0.009587	-0.212857	7.524178
MSCI	2000	0.000280	0.010147	0.035867	6.478511

Results in table 5 show that all the considered series are asymmetric with skweness coefficients significantly different from zero and are fat tailed as we can see from the large and positive values of kurtosis.

Daily sotck index returns are presented in Fig 1 and show a random fluctuations around zero and seems to be stationary. Fig 2 highlight the discrepancies of the data's empirical distributions and its divergence from the normal one. Plots of the density functions as well as the normal density functions are given in fig 3 for comparaison. The distributions are negatively skwed and exhibit an excess kurtosis.

7.2 Estimation of an ARFIMA on the data

For every series of data, we fit an ARFIMA(p,d,q) using the exact maximum likelihood methods. We allow p and q to take values in $\{1,2,3\}$, then for every index returns we fit 16 ARFIMA models and we select the ARFIMA model which, the best, fits the data according to AIC and SIC criteria.

Table 6: selected ARFIMA models for each index returns series.

	AIC	SIC
SP500	ARFIMA(3, 0.0286403,2)	ARFIMA(0, 0.0924989,0)
	t_d = 0.850	t_d = 4.89
	LM=6542.02222	LV= 6536.74005
FTSE	ARFIMA(3, 0.0340712,3)	ARFIMA(0, 0.0576491,0)
	t_d = 1.26	t_d= 3.23
	LM=6369.74445	LM= 6362.60815
CAC40	ARFIMA(2, 0.0183910,2)	ARFIMA(0, 0.0220006,0)
	t_d= 1.02	t_d= 1.22
	LM= 6197.4392	LM= 6192.48471
DAX	ARFIMA(2, 0.0516471,0)	ARFIMA(0, -0.0105515,1)
	t_d= 1.49	t_d= -0.445
	LM= 7232.85604	LM= 7230.68557
MIB	ARFIMA(1, 0.0404175,3)	ARFIMA(1, -0.0181351,2)
	t_d= 1.04	t_d= -0.736
	LM=5730.03015	LM=5727.16003
nikkei	ARFIMA(3, -0.00860522,2)	ARFIMA(0, -0.00239766,0)
	t_d= -0.310	t_d= -0.136
	LM=5785.00769	LM=5772.30098
TSE	ARFIMA(2, 0.0166063,1)	ARFIMA(0, 0.0924989,0)
	t_d= 0.501	t_d= 4.89
	LM= 6544.73435	LM= 6536.74005
MSCI	ARFIMA(0, -0.0300671,1)	ARFIMA(1, -0.0851530,0)
	t_d= -2.30	t_d= -1.22
	LM= 6233.51166	LM= 6234.68342

For every stock index the first line contains orders of the AR and MA parts in the ARFIMA model as well as the value of d in the middle. The second line contains the t-statistic and the last line gives values of the log-likelihood. Fig 4 contains the autocorrelograms of daily returns indexes, where the maximum lag is 600. For the CAC40 and Nikkei returns series only the lowest autocorrelations are statistically significant. For the other series, not only the lowest autocorrelations are statistically significant but also autocorrelations of higher order. This graphical behaviour of data autocorrelations make possible the existance of a long-run component on this data.

Results in table 6, show that four of the eight series studied here reveal long memory. In fact for those series the value of estimated d is significantly different from zero, it is positive for SP500, FTSE and Canada and negative for the world market.

Table 7 : Threshold test on the MSCI stock market returns (AR part)

serie	SupLM	AveLM	ExpLM	Threshold
MSCI	0.01600	0.00200	0.0000	-0.01975

Among the series exhibiting long memory, only the MSCI series has autoregressive order different from zeros, we apply the threshold test on its returns series in order to detect a potential asymmetric short-run behaviour governing its movements.

Table 7 reports the results of the three test statistics. The null hypothesis of linearity is strongly rejected in the 5%. The world index returns series is governed by an asymmetric behaviour and a threshold effect. For this series, the reponse to a shock occuring at time t-1 is different according to whether the shock is larger or less than the threshold value.

Table 8 : Threshold on the G7 stock market returns (d part)

serie	supLM	aveLM	expLM	Threshold
SP500	0.00000	0.00000	0.00000	-0.01887
FTSE	0.00400	0.01100	0.02300	-0.01805
TSE	0.00000	0.00000	0.00000	-0.01887
MSCI	0.05300	0.01800	0.00700	-0.02307

Table 8 shows that a threshold effect exists clearly in the long range parameter of the SP500, FTSE and TSE series. For the MSCI series, the SupLM statistic shows no threshold effect, the AveLM and SupLM statistics show the existance of a threshold effect. According to the sample size, the supLM has a better size when the number of observations is 2000 so we could trust the SupLM statistic and say that the MSCI returns series doesen't exhibit a threshold effect. This decision is confirmed by a forecasting study which is not reported here.

Let focus on the SP500 series. For this series, we fit an ARFIMA (0,d,0) model for every regime.

Table 9: ARFIMA (0,d,0) for every regime in the SP500

	parameter values	t-stat
regime 1		
d	-0.355620	-3.77
intercept	0.00115185	2.21
regime 2		
d	-0.0311474	-2.15
intercept	0.000295695	1.84

According to table 9, the t-statistics related to the d parameter are significant so the SP500 series is governed by two regimes, each one is exhibiting a long memory component. It is interessant at this stage to ask for the usefulness of our Threshold ARFIMA model.

8 Forecasting performance of Threshold ARFIMA models

In this step we assess, via a forecasting study, the performance of our threshold ARFIMA model over the symmetric ARFIMA.

8.1 Forecasting ARFIMA models

Let $\{X_t\}$ be an invertible[6] process following an ARFIMA (p,d,q) model: Assuming invertibility allows writing:

$$\varepsilon_t = \sum_{j=0}^{\infty} \pi_j L^j = \Phi(L)\Theta^{-1}(L)(1-L)^d \tag{19}$$

Generalisation of Brockwell and Davies [1991, p.183] to include ARFIMA models yields to the following process:

$$\widetilde{X}_{n+h} = -\sum_{j=1}^{\infty} \pi_j \widetilde{X}_{n+h-j}{}^{7} \tag{20}$$

For the SP500 series, the performance of the threshold ARFIMA model is measured by the RMSE (Root Mean Squared Errors),

$$RMSE = \sqrt{\frac{1}{N_j} \sum_{i=0}^{N_j-1} [\widetilde{A}(t+i+j) - A(t+i+j)]} \tag{21}$$

The model is perfect if $\widetilde{A}(t+i+k) = A(t+i+k)$,in other words if RMSE is null. In emprical studies the model is not perfect. The smaller the RMSE, the better is the performance of the model.

The performance of the thereshold ARFIMA model can be measured also by the MAE criterion (Mean Absolute Error),

$$MAE = \frac{\sum_{i=0}^{N_j-1} \left| \widetilde{A}(t+i+j) - A(t+i+j) \right|}{N_j} \tag{22}$$

[6] a process is invertible of $\theta(L) \neq 0$ for $|L| \leq 1$
[7] truncation lag is n-1+h (see Brockwell and davies [1991]

8.2 Empirical results

Results for the SP500 series are collected in the following table :

Table 10: forecasting accuracy test for the SP500 series

horizon	criterion	symmetric ARFIMA	Asymmetric ARFIMA
1 day	RMSE	0.0064187458	0.0064546843
	MAE	0.0048350350	0.0048187303
5 days	RMSE	0.0079335405	0.0079355828
	MAE	0.0063839825	0.0063284674
20 days	RMSE	0.0066284811	0.0065480726
	MAE	0.0051187214	0.0049141430

Table 10 shows that the asymmetric ARFIMA model does better than the symmetric one according to the MAE criterion for all horizons. On the other hand, according to the RMSE criterion, the asymmetric ARFIMA model perform better than the symmetric one for the horizon 20 days (1 month) but worse when the horizon is one day or five days. It should be noted that the RMSE criterion is, according to MEESE and ROGOFF [1983], a bad criterion if the studied series is governed by a stable paretien non normal process with an infinite variance. So it is better to refer to the MAE criterion. The authors announce that the MAE is also a useful criterion if the studied series is fat tailed, even if its variance is finite.

The superiority of the asymmetric ARFIMA (0,d,0) model fitted to the SP500 series over the symmetric ARFIMA (0,d,0) could be explained by the fact that the ARFIMA (0,d,0) is the best model selected in every regime of the SP500 series. In fact, in the first regime, the ARFIMA (0,d,0) is selected according to the AIC and SIC criteria and in the second regime, according to the SIC criterion.

9 Conclusion

This paper has considered testing for threshold in the fractional integrated ARMA models. Tests were driven in the autoregressive as well as the long run parameter. As an application, after selecting the best ARFIMA (p,d,q) model for the G7 and the world market indexes returns, we have tested for short range and long range asymmetry, respectively, on the series exhibiting long memory. We found powerful results showing that the MSCI return series is gpverned by an ARFIMA (3,d,2) with an asymmetric autoregressive component. A more important result shows that the SP500 daily return series presents an asymmetric long-run behaviour.

In Pratice, this finding permits to have more accurate forecasting and allow decision makers to take better strategic decisions because many financial fields, as pricing and financial inetermediation, are correlated, positevely or negatively, with the financial markets.

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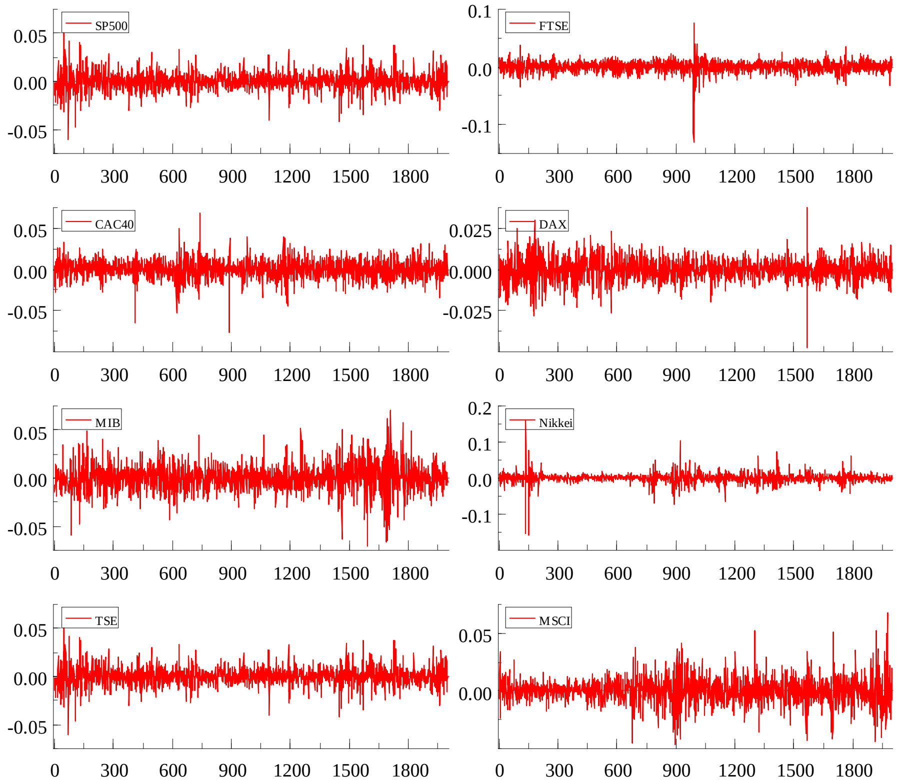
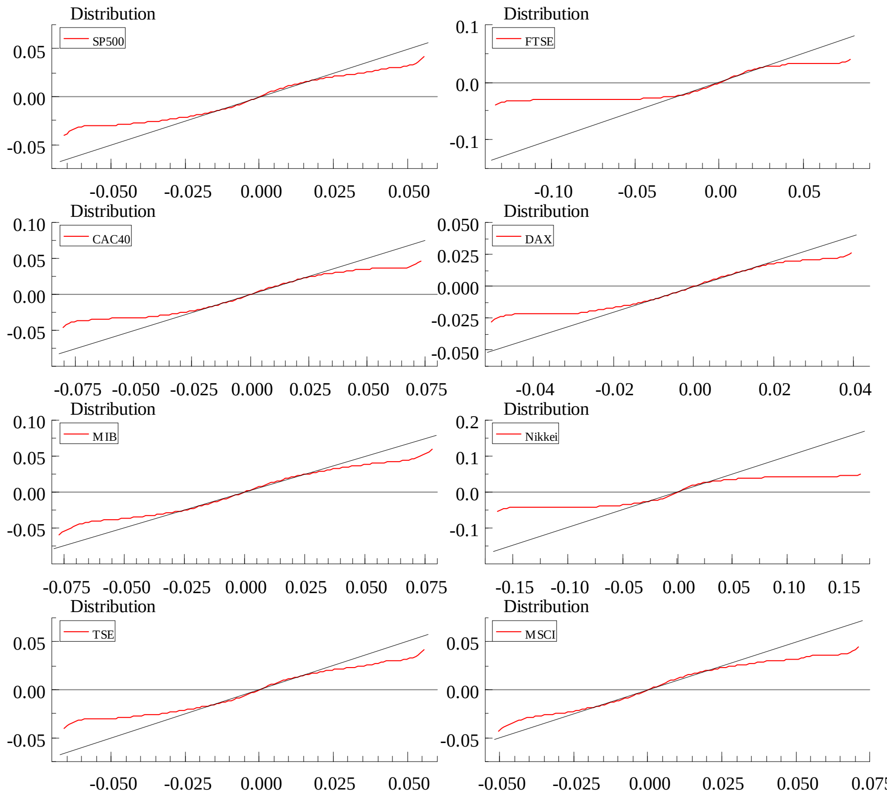
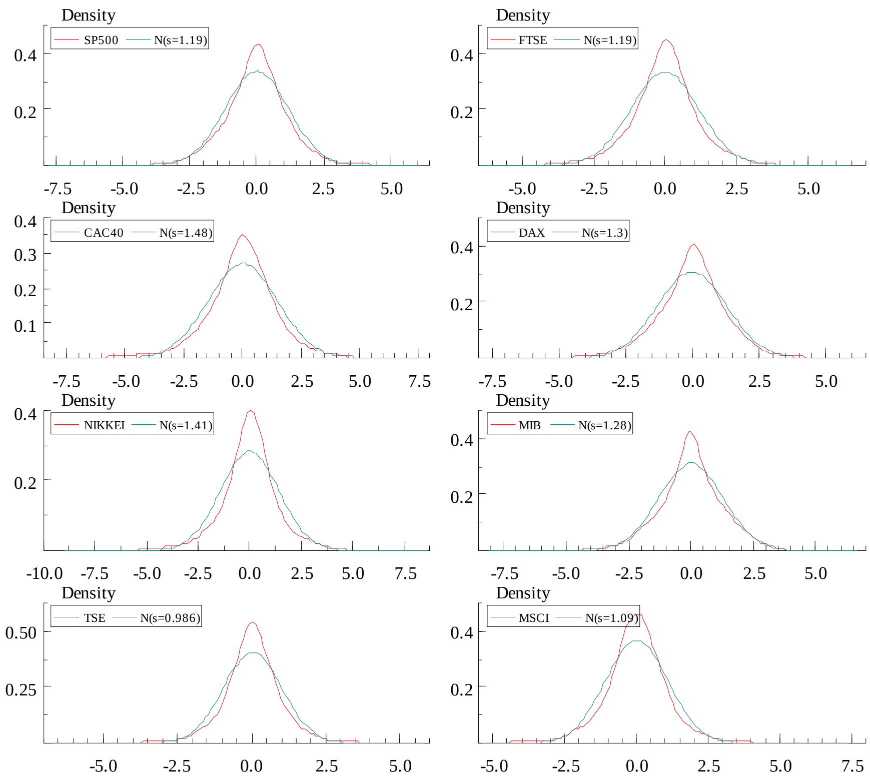


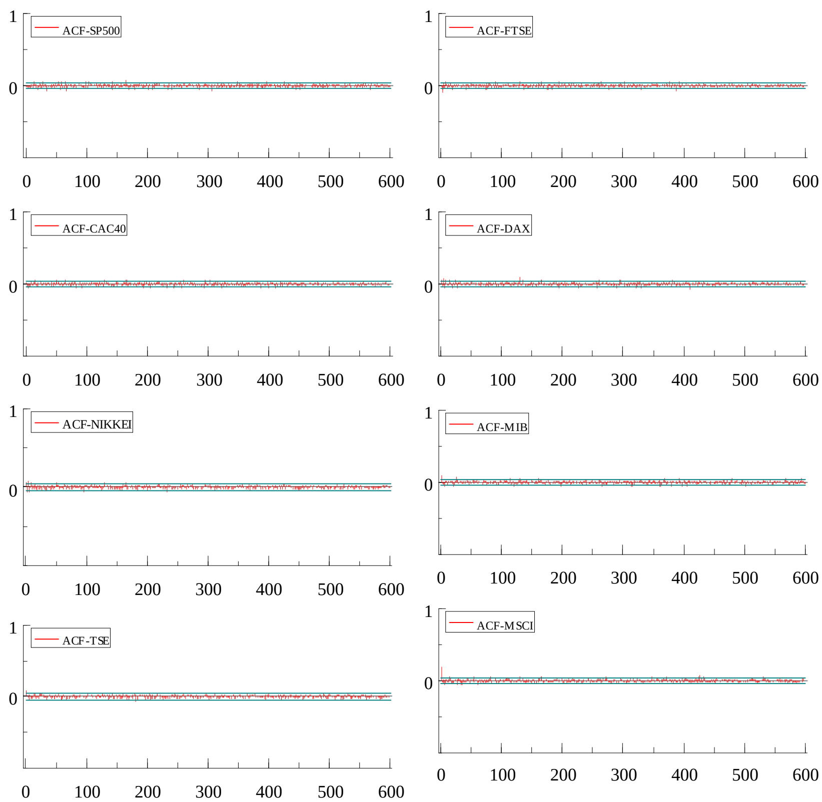
Daily stock Indexes market returns



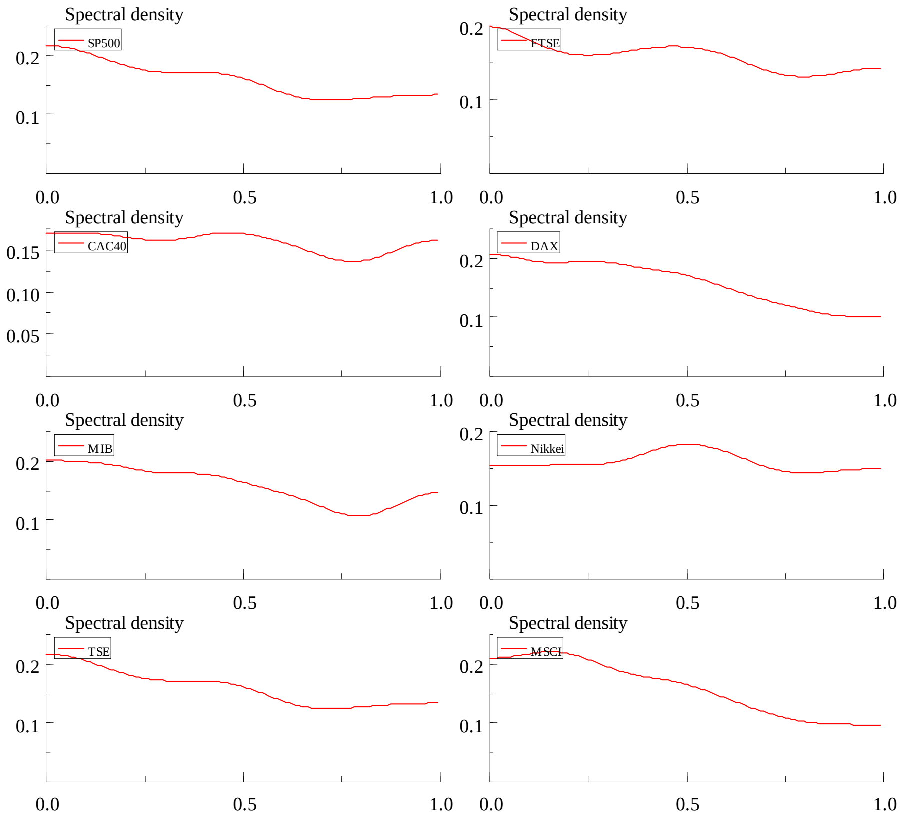
Stock Market Returns Distributions



Densities against normal



Autocorrelograms of daily stock markets indexes returns



Spectral Densities of the Stock Market Returns