(Un)naturally Low?*
Sequential Monte Carlo Tracking of the Natural Rate of Interest

Marco J. Lombardi† Silvia Sgherri‡
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Abstract
Have interest rates been held "too low" in relation to the natural rate of interest over recent years? Using a dynamic optimizing business cycle model satisfying the natural rate hypothesis and generalized to allow for nonstationarity in the underlying stochastic processes, this paper attempts a real-time evaluation of the US monetary policy stance, while ensuring consistency between the specification of price adjustments and the evolution of the economy under flexible prices. To do this, the model's likelihood function is evaluated using a Sequential Monte Carlo algorithm, providing joint estimates of the structural parameters and the unobservable, time-varying, nonstationary state variables. As a result, a detailed tracking of the whole distribution of the natural rate can be obtained, enabling a thorough assessment of half century of US monetary policy.

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†University of Pisa and European Central Bank. Email: mjl@ec.unipi.it.
‡European Central Bank and International Monetary Fund. Email: silvia.sgherri@ecb.int.
1 Introduction

How can policymakers and market participants assess whether monetary policy is expansionary, restrictive or neutral? Economists generally tend to focus on the real interest rate and compare it with some equilibrium, or natural, level which is expected to prevail over the long run. The concept of natural interest rate was devised more than a century ago by a Swedish economist, Knut Wicksell, and further developed by Austrian monetary business cycle theorists, such as Friedrich Hayek. Put simply, the natural (real) rate of interest is the rate that keeps output at its potential and inflation stable, once any shocks to the economy have played out. According to this concept, one can thus gauge the stance of, say, the Fed’s monetary policy by comparing the actual level of the federal fund rate with the natural rate. If the rate set by the central bank (the cost of capital) is lower than the natural rate, the economy and inflation would be expected to accelerate, as there will be excessive investment and borrowing, while households will not save enough. Conversely, if the Fed keeps interest rates above the natural rate, policy would rein in the economy and inflation would eventually slow. Yet, to make life tricky for both policymakers and market participants, the natural rate of interest is not observable and may vary over time, in line with changes in the rate of return on capital or households’ rate of time preference.

There are indeed many reasons to think that the natural (real) rate may have risen over the last decade. The surge in productivity growth owing to the information technology revolution is likely to have boosted expectations about future profits and investment opportunities, moving upwards the demand curve for investment funds, ceteris paribus. The entry of emerging countries’ cheap and unskilled labor force into the global economy might have lifted the labor-to-capital ratio, further increasing the worldwide expected return on capital. Meanwhile, households in industrial countries seem to have become more impatient and decided that they need to save less than they used to, on the belief that rising asset and house prices will provide them with adequate resources to finance their retirement. As a result, the so-called IS (Investment=Saving) curve—the equilibrium line showing the negative relationship between spending and real interest rate—may have shifted rightwards, implying a higher natural rate of interest for any level of potential output.

Conversely, structural forces—such as enhanced competition from emerging markets, deregulation, and faster productivity growth—may have helped to hold down (the equilibrium level of) inflation, whilst central banks’ success in taming inflation (around such an equilibrium level) has also reduced inflationary expectations. This has meant that monetary policy can
now be eased more freely to deal with economic downturns and bouts of financial instability, as long as inflation remains subdued. Certainly, after the US stockmarket bubble burst, the Fed seems to have successfully managed to reduce economic volatility in the short term (resulting in the mildest US recession ever!) by reducing short-term rates to a 45-year low of 1 percent, while effectively convincing the bond markets that rates would have been kept so low for a considerable period. The question is to determine whether in 2001-03 the Fed has cut interest rates by "too much" and left them low for "too long" in relation to the expected return on capital. Worringly, the exceptionally low cost of capital, at precisely a time when the natural rate of interest has likely increased, might have encouraged excessive borrowing, allowing financial imbalances to build up. Aware of these risks, as inflation has started to edge up, the Fed has started to lift interest rates gradually. But have the Fed's 16 rate hikes completely removed policy accommodation—thereby allowing the real Fed fund rate to rise again to the natural rate of interest—or are further hikes to be expected?

The relevance of these questions gives an idea of the central role played by time-varying equilibrium real rate of interest for policymaking. The problem of real-time estimation of such unobservable variable in the context of monetary policy analysis has hence received increasing attention in the literature. The findings have not always been satisfactory, though.

From an empirical perspective, a common result of quantitative studies in monetary economics is that real-time estimates of the time-varying natural rate of interest—like those of the equilibrium level of output, unemployment, and inflation—tend to be highly imprecise.\footnote{See, for instance, Laubach (2001), Larsen and McKeown (2002), Orphanides and Williams (2002), Laubach and Williams (2003), Basdevant, Björksten, and Karagedikli (2004), Cuaresma, Gnan, and Ritzberger-Grunenwald (2004), Mésonnier and Renne (2004), Sevillano and Simon (2004), Garnier and Wilhelmsen (2005).} Over small samples, trimming down uncertainty around time-varying estimates remains, indeed, very difficult. From a theoretical perspective, examining the behavior of the natural rate of interest within a dynamic optimizing sticky-price model of the business cycle raises exceptional challenges. To begin with, compliance with the natural rate hypothesis (NRH) becomes inescapable. As stressed by McCallum (1998) and, more recently, by Andrés \textit{et al.} (2005), while real business cycle models assuming fully flexible prices automatically satisfy the NRH—as actual output equals its natural level in each period—neo-Keynesian models allowing for price rigidities may entail a nonzero output gap in steady state which is positively related to the equilibrium rate of inflation. This would imply that, under price stickiness, monetary policy may enrich agents permanently compared to an economy in which prices are flexible—a feature which is clearly in contrast with the Wicksellian concept of natural interest rate. At the same
time, the definition of natural real rate of interest underpinning the Woodford’s (2003) "neo-Wicksellian" approach to price determination must account for possible permanent shifts in the steady-state rates of inflation and growth. Ensuring consistency between the specification of price adjustments and the underlying evolution of the economy under flexible prices appears especially crucial in system estimation. In this context, each change to the specification of price adjustments affects the likelihood-maximizing vector of structural parameters which—together with the dynamics and the distributions of shocks—determine the (unobservable) behavior of the economy under flexible prices.\(^2\)

Our paper attempts a real-time evaluation of the US monetary policy stance using a dynamic optimizing business cycle model, which (i) satisfies the NRH, (ii) ensures consistency between the specification of price adjustments and the evolution of the economy under flexible prices, and (iii) is generalized to allow for nonstationarity in the underlying stochastic processes. To put it in another way, the paper represents an original effort to take a DSGE model to non-detrended data: it tries to match business cycle and growth features of the data at the same time, by using model’s restrictions at business cycle frequencies while relaxing any co-trending implications at low frequencies.

To do this, the model’s likelihood function is evaluated using a Sequential Monte Carlo algorithm, providing joint estimates of the structural parameters and the unobservable, time-varying, nonstationary state variables. The idea beneath this approach is to represent any probability law by a large number of random samples, or particles, evolving over time on the basis of a simulation-based updating scheme, so that new observations are incorporated in the filter as they become available. In this way, the whole distribution of the natural rate can be obtained, enabling a thorough assessment of half century of US monetary policy.

The rest of the paper is organized as follows. Section 2 lays down a dynamic stochastic model consistent with the neo-Wicksellian approach to price determination. Section 3 presents the methodology adopted to evaluate the likelihood of such a model, estimate posterior densities for its parameters, and track down the time-varying distribution of its unobservable state variables. Section 4 discusses corresponding estimation and forecasting results, while Section 5 concludes the paper by providing an outlook on future work.

2 A Generalized "Neo-Wicksellian" Framework

Our structural model has a prototypical rational expectations specification, similar to the one used in Boivin and Giannoni (2003), Giannoni and Woodford (2005), and described in Woodford (2003). On the demand side, we take into account habit persistence in the level of aggregate expenditure, assuming that for each household $i$ in period $t$, utility depends not only on current expenditure, but also on the level of expenditure in the previous period. On the supply side, we allow for price indexation for firms that are not allowed to set their price optimally in a given period, in order to generate more realistic level of inflation inertia. The setup is then originally enriched by letting in exogenous non-stationary processes—describing (labor) productivity and target inflation dynamics, respectively—as well as idiosyncratic demand, supply, and policy shocks.

2.1 Demand Side

We assume a model economy producing a continuum of goods indexed by $j$ and populated by a continuum of households indexed by $i$, uniformly distributed over the $[0, 1]$ interval. By considering the limiting case of a cashless economy—and thereby abstracting from real balances—each household $i$ seeks to maximize the following infinite discounted sum of future utilities:

$$E_t \left\{ \sum_{\tau=t}^{\infty} \beta^{\tau-t} \varepsilon^u \left[ \frac{1}{1 - \sigma} \left( C^i_\tau - \eta C^i_{\tau-1} \right)^{1-\sigma} - \frac{1}{1 + \chi} \left( H^i_\tau (j) \right)^{1+\chi} \right] \right\}$$

(1)

where $C^i_\tau$ is an index of the household’s consumption of the differentiated goods supplied at $t$ and $H^i_\tau (j)$ denotes the amount of hours supplied by each household $i$ for the production of each good $j$. In addition, $\beta \in (0, 1)$ is the households’ discount factor, $\sigma$ is the coefficient of relative risk aversion of households or the inverse of the intertemporal elasticity of substitution, and $\chi$ represents the inverse of the elasticity of work effort with respect to the real wage. The parameter $0 \leq \eta \leq 1$ measures the degree of habit formation in consumption. Consumers’ utility hence depends positively on deviations of consumption $C^i_\tau$ from an existing stock $\eta C^i_{\tau-1}$, and negatively on the total labor supplied. Equation (1) also contains a shock to the discount rate that affects the intertemporal preferences of households, $\varepsilon^u$, distributed as a log-normal with mean equals to 1.

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3This extension has been recently proposed by Christian, Eichenbaum, and Evans (2005). Inflation indexation has been also used, for instance, in Smets and Wouters (2005), Milani (2005), and Andrés et al. (2005).
Households maximize their objective function (1) subject to an intertemporal budget constraint that is given by:

\[ b_t \frac{B^t}{P^t} = \frac{B^{t-1}}{P^{t-1}} + (W_i^t H_i^t + A_i^t) - C_i^t \]  

(2)

implying that households hold their financial wealth in the form of one-period securities, \( B_T \), having price \( b_T \). Total income—consisting of labor income and the net cash inflow from participating in state-contingent securities, i.e., \( Y_i = W_i^t H_i^t + A_i^t \)—is solely used for consumption purposes.\(^4\) Under the assumption of complete financial markets and efficient risk sharing, state-contingent securities would insure households against household-specific fluctuations in labor income, so that each household faces an identical intertemporal budget constraint and an identical marginal utility of total income. As a result, the superscript \( i \) can be omitted without loss of generality.

With habit formation, the maximization of the objective function (1) subject to the budget constraint (2) with respect to consumption can therefore be written as:

\[ \lambda_t = \varepsilon^u_t (C_t - \eta C_{t-1})^{-\sigma} - \eta \beta E_t \left[ \varepsilon^{u+1}_t (C_{t+1} - \eta C_t)^{-\sigma} \right] \]

(3)

where the marginal utility of real income at time \( t \) (i.e., the Lagrange multiplier for the budget constraint, \( \lambda_t \)) depends upon both today’s and tomorrow’s expected marginal utility of consumption, as well as preference shocks occurring at time \( t \) and \( t + 1 \). In addition, the marginal utility of income satisfies the first order condition for the optimizing consumer’s problem with respect to bonds holdings, yielding:

\[ \lambda_t = \beta E_t \left[ \lambda_{t+1} (1 + \eta i) \frac{P_t}{P_{t+1}} \right] \]

(4)

where \( (1 + \eta i) = \frac{1}{\delta} \) denotes the riskless one-period (gross) nominal rate of return on bonds and \( E_t \left[ \frac{P_{t+1}}{P_t} \right] = (1 + E_t \pi_{t+1}) \) defines the expected (gross) rate of inflation. Using (3) to substitute for the \( \lambda \)'s in (4), we can thus derive the generalized Euler equation in the presence of habit formation, namely:

\[ \frac{\varepsilon^u_t (C_t - \eta C_{t-1})^{-\sigma} - \eta \beta E_t \left[ \varepsilon^{u+1}_t (C_{t+1} - \eta C_t)^{-\sigma} \right]}{E_t \left[ \varepsilon^{u+1}_t (C_{t+1} - \eta C_t)^{-\sigma} \right]} - \eta \beta E_t \frac{\varepsilon^{u+2}_t (C_{t+2} - \eta C_{t+1})^{-\sigma}}{(1 + \eta \pi_{t+1})} = \frac{(1 + \eta i)}{(1 + E_t \pi_{t+1})} \]

(5)

\(^4\)Supposing that the government implements a Ricardian fiscal policy while levying lump sum taxes, fiscal policy can be ignored given that it has no effect on model aggregates.
Re-expressing each variable in (5) as a ratio to its own (time-varying) equilibrium level and using the equilibrium relation \( C_t = Y_t \) — where \( Y_t \) represents aggregate income in period \( t \) — it yields:

\[
\frac{\epsilon_t^u \left( \frac{Y_t}{Y^*_t} - \eta \frac{Y_{t-1}}{Y^*_{t-1}} \right)^{-\sigma} - \eta \beta E_t \left[ \epsilon_{t+1}^u \left( \frac{Y_{t+1}}{Y^*_t} - \eta \frac{Y_1}{Y^*_1} \right)^{-\sigma} \right]}{E_t \left[ \epsilon_{t+1}^u \left( \frac{Y_{t+1}}{Y^*_t} - \eta \frac{Y_{t-1}}{Y^*_{t-1}} \right)^{-\sigma} \right] - \eta \beta E_t \left[ \epsilon_{t+2}^u \left( \frac{Y_{t+2}}{Y^*_t} - \eta \frac{Y_{t+1}}{Y^*_{t+1}} \right)^{-\sigma} \right]} = \frac{(1 + \delta_t) (1 + E_t \pi^*_t)}{(1 + \delta^*_t) (1 + E_t \pi^*_t+1)} \quad (6)
\]

where starred variables denote equilibrium levels and \( \epsilon_t^u \) represents the ratio of the lognormally-distributed error \( \varepsilon_t^u \) to its expected value. The more familiar log-linearized approximation of (6) is given by:

\[
\bar{y}_t = E_t \bar{y}_{t+1} - [\sigma (1 - \beta)]^{-1} \left[ \bar{\delta}_t - E_t \bar{\pi}_{t+1} \right] + \left( \bar{\zeta}_t - \bar{\zeta}_{t+1} \right) \quad (7)
\]

where

\[
\bar{y}_t = (\bar{y}_t - \eta \bar{y}_{t-1}) - \eta \beta E_t (\bar{y}_{t+1} - \eta \bar{y}_t) \quad (8)
\]

\[
\bar{\zeta}_t = \zeta_t - \eta \beta E_t (\zeta_{t+1}^u) \quad (9)
\]

with \( \xi_t^u \) denoting normally distributed aggregate demand shocks and hatted variables indicating (log-)deviations from corresponding time-varying equilibrium levels, e.g.:

\[
\hat{y}_t = y_t - y_t^* \quad (10)
\]

\[
\hat{\gamma}_t = \gamma_t - \gamma_t^* \quad (11)
\]

\[
\hat{\pi}_t = \pi_t - \pi_t^* \quad (12)
\]

Note that as \( \eta \to 0 \), expression (6) collapses to the standard Euler equation for consumption, while expression (7) reduces to the well-known log-linear forward-looking IS curve, where the output gap is expressed in terms of deviations of actual output from its equilibrium path.

### 2.2 Supply Side

We further assume a continuum of monopolistic competitive firms indexed by \( j \) and uniformly distributed over the \([0, 1]\) interval. Each firm is a monopolistic supplier of good \( j \), which is produced according to the production technology \( Y_t(j) = Z_t H_t(j) \), where \( H_t(j) \) is labor
input and \( Z_t \) describes an exogenous nonstationary productivity process common to all firms. Indicating with \( g_t = \ln(G_{t-1}) \) a stochastic drift and with \( \zeta_t^\eta \) idiosyncratic innovations to such a drift, the log of the productivity process, \( z_t = \ln(Z_t) \), evolves according to:

\[
\ln(Z_t) = \ln(Z_{t-1}) + \ln(G_{t-1}) + \zeta_t^\eta \tag{13}
\]

\[
\ln(G_t) = \ln(G_{t-1}) + g_t \tag{14}
\]

thereby inducing a stochastic trend in aggregate output and consumption. It follows that our model economy evolves along a stochastic growth path, characterized by a stochastic drift, \( g_t \).

Capital is regarded as fixed, leaving labor as the only variable factor of production. Labor is hired in a perfectly competitive factor market, at the hourly real wage \( W_t \).

Firms face a common demand curve \( Y_t(j) = Y_t \left( \frac{p_t(j)}{P_t} \right)^{-\chi} \) for their products, where the parameter \( \chi \) represents the elasticity of demand for each good \( j \), \( Y_t = \int_0^1 Y_t(j) \frac{1}{1-\chi} \, dj \) is aggregate output and \( P_t \) is aggregate price, both taken as given. Hence, all firms face identical maximization problem and, if allowed to choose their own price, set a common price \( p_t^* \).

Following Calvo (1983), however, we suppose that only a fraction \( 0 < (1 - \alpha) < 1 \) of firms are allowed to change their price in a given period, while the remaining fraction \( \alpha \) simply adjusts prices according to an indexation rule. In contrast to baseline Calvo price setting, we postulate that those price setters who cannot reset their prices today, have their prices automatically raised by a percentage \( 0 < \gamma < 1 \), so that \( \gamma \) represents the degree of indexation to past inflation.

Under standard properties of the profit function of each supplier \( j \), and assuming that profits are discounted using a stochastic discount factor equal—on average—to \( \beta \), Woodford (2003) demonstrates that the resulting inflation dynamics can be log-linearized as:

\[
\hat{\pi}_t - \gamma \hat{\pi}_{t-1} = \beta E_t (\hat{\pi}_{t+1} - \gamma \hat{\pi}_t) + \mu \{ \omega \hat{y}_t + [(1 - \eta \beta) \sigma] \hat{y}_t \} + \zeta_t^\pi \tag{15}
\]

where

\[
\mu = \frac{(1 - \alpha)(1 - \alpha \beta)}{\alpha (1 - \chi \omega)} \tag{16}
\]

\( \zeta_t^\pi \) is a white noise exogenous aggregate supply shock, and \( \hat{y}_t, \hat{y}_t, \) and \( \hat{\pi}_t \) are defined exactly as in (8), (10), and (11), respectively.

Note that, in the presence of habit formation \( (\eta > 0) \), the log marginal utility of total real income entering the Phillips curve (15) is written as a linear function of \( \hat{y}_t \), rather than simply
as a linear function of $\tilde{y}_t$ (which would be the case for $\eta = 0$). Moreover, indexation to past inflation ($\gamma > 0$) in the form specified in (15) ensures that the NRH is also satisfied, albeit locally: on average, output equals its potential level for an exogenous steady-state inflation rate—meaning that dynamic indexation only describes temporary fluctuations of the economy around its time-varying equilibrium.

In the absence of nominal rigidities, the evolution of aggregate output over time is governed by the productivity process described by (13) and (14), the degrees of habit formation and market distortion, e.g.: 

$$\ln(Y^*_t) = [(1 - \eta \beta) \sigma]^{-1} \ln \left( \frac{X-1}{X} \right) + \ln(Z_t)$$

which defines the (unobservable) equilibrium level of output under flexible prices, $y^*_t$, that appears in the output gap (10). It should be stressed that in our modeling framework the concepts of "potential", "equilibrium", "steady-state" and "flexible-price" level of output coincide: they are hence interchangably denoted by $y^*_t$ and described by the same time-varying, unobservable, nonstationary process.

The corresponding flexible-price natural real rate of interest—the real interest rate in an equilibrium where rigidities are ruled out and the output gap is zero at all times—is defined as:

$$\ln (1 + r^*_t) = [(1 - \eta \beta) \sigma] \ln \left( \frac{\varepsilon^u E_t[Y^*_t]}{E_t [\varepsilon^u_{t+1} Y^*_t]} \right)$$

with

$$\ln (1 + r^*_t) \equiv \ln \left[ \frac{(1 + i^*_t)}{(1 + \pi^*_t+1)} \right]$$

We will examine the law of motion for the steady-state inflation rate upon description of the monetary policy.

### 2.3 Monetary Policy

Monetary policy is introduced in the model through a generalized Taylor rule with partial adjustment and time-varying target for inflation:

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5In the absence of habit formation, the aggregate dynamics of inflation with indexation would reduce to $\tilde{\pi}_t - \gamma \tilde{\pi}_{t-1} = \beta E_t (\tilde{\pi}_{t+1} - \gamma \tilde{\pi}_t) + \mu (\omega + \sigma) \tilde{y}_t + \varepsilon^\pi_t$, where $(\omega + \sigma^{-1})$ measures the elasticity of the average real marginal cost with respect to the aggregate output.
\[(i_t - i_t^*) = \rho (i_{t-1} - i_{t-1}^*) + (1 - \rho) [\varphi_{1t} (\pi_t - \pi_t^*) + \varphi_{2t} (y_t - y_t^*)] + \zeta_t^r \]  

(20)

where \(\rho\) denotes the degree of interest rate smoothing, \(\varphi_{1t}\) and \(\varphi_{2t}\) are the (time-varying) long-run feedback coefficients to deviations of inflation and output from their respective (time-varying) steady-state levels, while \(\zeta_t^r\) accounts for unanticipated deviations from the systematic monetary policy rule.\(^6\) A similar rule allows for shifts in the target for inflation, while it assumes that the systematic policy responses have also changed over time.

Exogenous shifts in the monetary policy target \(\pi_t^*\) are believed to follow a Markov-switching process such as:

\[\pi_t^* = \pi_{t-1}^* + k_\pi \ast \epsilon_t\]  

(21)

whose transition matrix is given by:

\[P(\epsilon_t = 1) = p_u\]  
\[P(\epsilon_t = -1) = p_d\]  
\[P(\epsilon_t = 0) = 1 - p_u - p_d\]  

(22) (23) (24)

given \(k_\pi = 0.25\%\).

The idea here is to capture true shifts in the objective of the central bank, under the presumption that the central bank target is revised discretely over time with a step of \(\frac{1}{4}\) of a percentage point. This is a relatively new approach in the literature, as in most cases the target inflation is assumed to follow a Gaussian random walk (see, for example, Amisano and Tristani (2006), Smets and Wouters (2004)).\(^7\) Clearly, the fact that the transition matrix has only three states might be viewed as a restriction, but this limitation can be waived at a relatively negligible cost.

### 2.4 Model Solution

Equations (7) to (24) form a linear rational expectation system in the state variables

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\(^6\) An alternative specification of the monetary rule will be considered, according to which deviations of the policy rate from equilibrium are allowed to adjust for deviations of inflation and output growth from their equilibrium values.

\(^7\) We think of our Markov switching process as a quite realistic representation of policymakers’ behavior, as it seems unlikely that, at a certain time, a monetary authority might lower its inflation target by, say, 0.003441%. Nonetheless, the model has been also estimated using a random walk inflation target. Results are available upon request.
\[
x_t = \begin{bmatrix} \hat{y}_t & \hat{\pi}_t & \hat{\gamma}_t & y_t^* & g_t & \pi_t^* & r_t^* & i_t \end{bmatrix}^\prime,
\]
driven by innovations \( \zeta_t = \begin{bmatrix} \zeta_t^u & \zeta_t^g & \zeta_t^\psi & \zeta_t^\omega \end{bmatrix}^\prime \). This rational expectation system has to be solved and cast in state space before the DSGE model can be estimated.\(^8\) The solution of the rational expectation system hence takes the form:

\[
x_t = F(x_{t-1}, \zeta_t; \theta),
\]

with \( F \) being a general function of the vector of structural parameters of our DSGE model, \( \theta \), defined as:

\[
\theta = \begin{bmatrix} \eta & \sigma^{-1} & \gamma & \alpha & \rho & p_u & p_d & \sigma_{\zeta^u} & \sigma_{\zeta^g} & \sigma_{\zeta^\psi} & \sigma_{\zeta^\omega} \end{bmatrix}^\prime,
\]

where \( \sigma_{\zeta^u}, \sigma_{\zeta^g}, \sigma_{\zeta^\psi}, \) and \( \sigma_{\zeta^\omega} \) are the standard deviations of the shocks to the (stochastic) productivity growth rate, to aggregate demand, aggregate supply, and monetary policy.\(^9\)

Depending on the parameterization of the structural model, there are three possibilities: no stable rational expectation solution exists; the stable solution is unique (determinacy); or there are multiple stable solutions (indeterminacy). We will focus on the case of determinacy and restrict the parameter space accordingly. For linear models like the one at hand, several solution algorithms are available.\(^10\) For the subsequent analysis, we use Klein’s (2000) generalized Schur form to solve our linearized DSGE model (7) to (24). Within a nonlinear framework, the identification of structural parameters becomes challenging given the nonlinear mapping from the structural form of the DSGE model into its state-space representation. In particular, under these conditions, first-order approximations may not suffice any longer, and higher-order refinement have to be brought into play.\(^11\)

The model is completed by defining a set of measurement equations that relate the elements of \( x_t \) to a set of observable variables, \( y_t = \begin{bmatrix} y_t & \pi_t & i_t \end{bmatrix}^\prime \), comprising quarterly data over the

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\(^8\)Even though the model can be re-written in terms of deviations from equilibrium, we do not induce stationarity by constructing a log-linear approximation of the model around the steady-state for the detrended variables (see, for instance, Altig et al. (2002)). Instead, our model is taken directly to non-detrended data without imposing any co-trending restrictions, which are hardly supported by empirical evidence (Canova et al. (1994), Del Negro et al. (2004)).

\(^9\)The values of the parameters \( \beta, \chi, \) and \( \omega \) are not estimated. They are fixed to 0.99, 7.52, and 0.989, respectively, in line with Giannoni and Woodford (2003) and Milani (2005).


period 1957:I to 2006:I on real GDP, annualized quarterly rate of change of the CPI less food and energy, and the annualized federal funds rate.\textsuperscript{12} Namely:

\begin{equation}
y_t = G(x_t, \zeta_t; \theta),
\end{equation}

The general state-space model (25)-(26) forms the basis for the likelihood estimation of the structural parameters \(\theta\), which—together with the distribution of the shocks \(\zeta_t\)—determine the underlying behavior of the economy under flexible prices.

\subsection*{2.5 Prior Specification}

One of the hardest task in implementing Bayesian estimation is how to specify sensible priors about the deep parameters of the model. While, in principle, priors should reflect strong beliefs about the validity of underlying microeconomic theory, in practice most priors are fine-tuned to match important properties of the aggregate data.

Table 1 summarizes information about our priors over the vector of structural parameters \(\theta\). For convenience, all parameters are assumed \textit{a priori} to be independent.

The habit and indexation parameters, \(\eta\) and \(\gamma\), are assumed to follow beta distributions with mean 0.5 and 0.9, and standard deviations 0.3 and 0.05, respectively. The prior for the coefficient \(\alpha\)—governing the degree of price stickiness—is chosen based on micro-evidence on price setting behavior, implying re-optimization of prices approximately every quarter (for instance, Bils and Klenow (2004)). The priors for the degree of interest smoothing in the monetary policy rule, \(\rho\), is centered around 0.9, while the prior means of the (time-varying) feedback coefficients, \(\varphi_{1,0}\) and \(\varphi_{2,0}\), are chosen to match values typically associated with the Taylor rule. To guarantee values in \(\mathbb{R}^+\), we postulate lognormal distributions for the standard deviation of shocks, \(\sigma_{\zeta}\), \(\sigma_{\zeta^*}\), and \(\sigma_{\zeta^*}\), whereas for the intertemporal elasticity of substitution, \(\sigma^{-1}\), we assume a gamma distribution with mean equal to [0,1]. Finally, since we do not have strong prior beliefs for \(p_d\) and \(p_u\)—lacking previous estimates in the literature—we prefer to represent them through diffuse beta distributions, implying generally low probability of switching, with \(p_d\) slightly higher than \(p_u\).

As for the starting values of the state variables, by and large they are chosen based on information at the beginning of the sample. In particular, the prior mean of the year-on-year rate of productivity, \(\ln(G_0)\), is 3.7 percent and that of \(\ln(Y_0^*)\) is 7.8. The prior for the inflation

\textsuperscript{12}The series were obtained from FRED, the database of the Federal Reserve Bank of Saint Louis. An alternative specification of the model has been estimated using headline inflation, rather than core inflation. Corresponding results are also discussed below.
target is centered at 3.25 percent. Finally, given the prior mean values of the structural parameters \( \theta \), the prior mean of the (real) natural rate of interest is chosen at 2 percent to satisfy (18).

3 Likelihood Evaluation and Bayesian Estimation

To fit the solved DSGE model to the vector \( y_t \) of observable variable, we employ a Bayesian estimation technique based on the likelihood function generated by the structural system of equations. Precisely, the model’s likelihood function is evaluated using a Sequential Monte Carlo algorithm—or particle filter—providing joint estimates of the structural parameters and the unobservable, time-varying, nonstationary state variables.

To our knowledge, this is a very innovative way to bring DSGE models to data. Up to now, in the setting of DSGE models, particle filters have been used to construct the likelihood, but not to perform parameter estimation. This is due to the unendurable computational burden that is needed to solve the model for each particle and for each time stamp. We propose an indirect way of proceeding which yields parameter estimates directly from the filtering procedure, through a single solution of the DSGE model for each particle.

Our idea is to impose priors on the structural parameters of our DSGE model, solve it, estimate the corresponding state-space system using the particle filter recursion, and map back the posterior distribution of the structural rational expectation model’s parameters. Specifically:

- Given the set of overidentifying restrictions implied by the model, we impose priors on the vector \( \theta \) and map the structural parameter space into the parameter space of the state-space model, say \( \phi \);
- The particle filter is then used for inference on the state-space model, where underlying nonstationary processes and static parameters are treated as state variables;
- Each set of \( M \) draws from the filtering distribution of the static parameters \( \phi \) is then mapped back into the parameter space of the structural model to get the posterior distribution of \( \theta \).

The idea of mapping one parameter space into another is somewhat similar to that of indirect inference (Smith (1993), Christiano et al. (2005)), although further work is currently in progress to investigate this parallel. From a technical standpoint, the mapping employed in our analysis implies a direct knowledge of the binding function. Contrary to plain indirect
approaches, however, no optimization algorithm is called for by the filtering methodology adopted here, as the estimates of the state-space model do already enjoy the necessary statistical properties. If a unique stable solution exists, the state-space model is derived univocally from the structural economic model: we can thus reasonably expect the misspecification bias to be relatively small.

Details of the Bayesian estimation procedure adopted in this paper are provided in the following subsections.

3.1 A Sequential Monte Carlo Approach

The particle filtering approach is an extremely powerful framework for inference in state-space model (Doucet et al. (2000)). Although these methods have originated in the engineering literature (Gordon et al. (1993), Kitagawa (1996)), they have recently been employed also for financial (Pitt and Shephard (1999)) and macroeconomic applications (Fernández-Villaverde and Rubio-Ramírez (2004a, 2004b), An and Schorfheide (2005), Amisano and Tristani (2006)).

The idea underlying this approach is to represent the distribution of the state variables by a Monte Carlo approximation constructed via a large number of random samples, or particles, evolving over time on the basis of a simulation-based updating scheme. In this way, new observations are processed by the filter as they become available. Each particle is assigned a weight, which is updated recursively.

Given the state-space model defined by (25)-(26), the filtering problem lies in the determination of $p(x_t|y_{1:t}, \phi)$. This can be performed in two steps:

1. Projection:

$$p(x_{t+1}|y_{1:t}, \phi) = \int p(x_{t+1}|x_t, \phi)p(x_t|y_{1:t}, \phi)dx_t$$

2. Updating:

$$p(x_{t+1}|y_{1:t+1}, \phi) = \frac{p(x_{t+1}|y_{1:t}, \phi)p(y_{1:t+1}|x_{t+1}, \phi)}{p(y_{1:t+1}|y_{1:t}, \phi)}$$  \hspace{1cm} (27)

$$p(y_{t+1}|y_{1:t}, \phi) = \int p(x_{t+1}|y_{1:t}, \phi)p(y_{1:t}|x_{t+1}, \phi)dx_{t+1}$$  \hspace{1cm} (28)

Under the assumption that $G$ and $F$ in (25)-(26) are linear functions and the shocks are Gaussian, a closed-form solution to the filtering problem is yielded by the Kalman filter. With the presence of nonlinearities and deviations from Gaussianity, things become more complex and approximated solutions are needed. One possibility is to linearize the model by means of Taylor series expansions. This approach is sometimes referred to as the Extended Kalman Filter.
(Harvey (1990)). Conditions for convergence are often quite difficult to establish, however, and in what follows we will concentrate on a simulation-based approach to filtering which has the very useful feature to be extremely general.

The idea of a Sequential Monte Carlo filter—as anticipated—is to represent the distributions of interest (and therefore its moments and quantiles) by Monte Carlo approximations, namely by a set of $M$ particles. Let suppose to be at time $t-1$ and generate $M$ draws from

$$p(x_t | x_{t-1}, \phi),$$

thereby plainly simulating from the system of state equations. Next, let assign to each particle $m = 1, \ldots, M$ a weight proportional to its likelihood:

$$w_{t,m} = p(y_t | x_t, \phi).$$

Weights can be then normalized as:

$$\tilde{w}_{t,m} = \frac{w_{t,m}}{\sum_{m=1}^{M} w_{t,m}},$$

The weights can thus be used to compute Monte Carlo integrals via importance sampling.

It turns out that, after a certain number of steps, the filter might eventually degenerate, i.e. assign all the weight to a single particle—namely the one relatively most likely with respect to the observed data. In order to avoid this, particles can be resampled: after having produced a set of particles and assigned to each one an appropriate weight, we associate to each particle $m$ a number of offspring $O_m$ such that $\sum_{m=1}^{M} O_m = M$. After this selection step, offspring particles replace the original particles and the importance weights are reset to $1/M$, so that the set of particles can be thought of as a random sample. The resampling step can be implemented at every time interval (Gordon, Salmond and Smith, 1993), or it can be employed whenever the set of particles crosses a certain degeneracy threshold. A measure of degeneracy of the algorithm is the effective sample size (Liu and Chen, 1998), defined as:

$$M_e = \frac{M}{1 + \text{Var}(w_t)}.$$  

This quantity can be estimated by:

$$\tilde{M}_e = \frac{1}{\sum_{m=1}^{M} \tilde{w}_{t,m}^2};$$

when $\tilde{M}_e$ drops below a certain threshold, the resampling takes place.

In our analysis, we compute the resampling in a computationally straightforward manner. Specifically, the number of offspring is taken to be proportional to the importance weight and
generated via simulation of a set $U$ of $M$ uniformly distributed random variables on $[0, 1]$, by using the cumulated sum of the normalized weights

$$q_m = \sum_{j=1}^{m} \tilde{w}_{t,j},$$

and then setting $O_m$ equal to the number of points in $U$ that fall between $q_{m-1}$ and $q_m$.

### 3.2 Static Parameters Estimates

Particle filters are, indeed, filters, and they were not designed to perform parameter estimation. In many cases, they can be used to obtain the likelihood function, which is then maximized with respect to the parameters via standard optimization techniques. This is the approach that has been followed up to now in the field of DSGE models.

To be sure, it is also possible to employ a particle filter to perform parameter estimation. To do this, one can simply pretend that static parameters are indeed time-varying states, by adding a noise term at each time interval. The problem however is that, by doing so, relevant information is "thrown away" and additional variability generated. Liu and West (2001) propose a method to quantify such a loss of information, by allowing for an artificial parameter evolution scheme immune to this problem. Introducing artificial parameter evolution is equivalent to consider a model in which $\phi$ is replaced by its time-varying analog $\phi_t$, which evolves according to:

$$\phi_t = \phi_{t-1} + \kappa_t, \quad \kappa_t \sim N(0, \xi_t),$$

where $\xi_t$ is a diagonal matrix.

In a situation in which $\phi$ is fixed, the posterior distribution $p(\phi|y_{1:t})$ can be characterized by its Monte Carlo mean and variance $\bar{\phi}_t$ and $s^2_t$, with $s^2_t$ denoting the vector of empirical variances of each element in $\phi$. It is immediate to observe that, in the case of artificial parameter evolution, the Monte Carlo variance increases to $s^2_t + \xi_t$. In fact, the Monte Carlo approximation can be expressed as kernel smoothed density of the particles as:

$$p(\phi|y_{1:t}) \approx \sum_{j=1}^{M} \kappa^{(j)}_t N\left(\phi_{t+1}^{(j)}|\phi_t^{(j)}, \xi_t\right), \quad (29)$$

whereas the target variance $s^2_t$ can be expressed as:

$$s^2_t = s^2_{t-1} + \xi_t + 2\text{Cov}(\phi_{t-1}, \kappa_t).$$

By choosing

$$\text{Cov}(\phi_{t-1}, \kappa_t) = -\frac{\xi_t}{2}$$
the loss of information can be easily avoided. A simple way to achieved this is to consider

\[ \xi_t = s_t^2 \left( \frac{1}{\delta} - 1 \right), \]

where \( \delta \) is a discount factor in \((0, 1]\). If we define \( d = \frac{3\delta - 1}{2\delta} \), the conditional density evolution becomes:

\[ p(\phi_{t+1}|\phi_t) \sim N \left( \phi_{t+1}|d\phi_t + (1 - d)/\xi_f, h^2 \xi_t^2 \right), \]

where

\[ h^2 = 1 - d^2 = 1 - \left( \frac{3\delta - 1}{2\delta} \right)^2, \]

so that sampling from (30) is equivalent to sampling from a kernel smoothed density in which the smoothing parameter \( h \) is controlled via the discount factor \( \delta \).

Clearly, the choice of the number of particles \( M \) affects both the approximation of the prediction error distribution and the performance of the resampling algorithm. Our choice of 30,000 particles has been driven, on the one hand, by the need to have enough particles to capture the tails of \( p(\phi|y_{1:t}) \), on the other hand, by the evidence that — after a certain threshold — gains in noise reduction from the use of more particles become very small (Fernández-Villaverde and Rubio-Ramírez (2004b), Lombardi and Godsill (2005)).

To wrap up, all the necessary information about the state-space model parameters \( \phi \) is summarized by the Monte Carlo approximation (29) of \( p(\phi|y_{1:t}) \), generated by \( M \) particles. Each particle is hence mapped back into the parameter space of the structural model, enabling us to devise Monte Carlo approximations to the posterior probability distributions of the structural parameters collected in \( \theta \).

### 3.3 Out-Of-Sample Forecasts

The sequential nature of the particle filter lends itself particularly well to perform out-of-sample forecasting of both state and observable variables. Indeed, the goal of the projection step is to construct random samples from the Monte Carlo approximation of the forecast distribution \( x_{t+1} \), computed using the posterior distribution of the state-space model parameters \( \phi \), e.g.:

\[ p(x_{t+1}|x_t, \phi). \]

Each random sample \( x_{t+1} \) from the projected (or forecast) distribution is then turned into a random sample from the filtered distribution as weights are assigned and resampling is perf-

\[ ^{13} \text{Liu and West (2001) suggest to choose a value of } \delta \text{ around 0.95-0.99.} \]

\[ ^{14} \text{Please note that whenever we refer to the out-of-sample performance of our DSGE model, we always consider the state-space representation of the DSGE model, and not its VAR approximation.} \]
formed. In order to obtain multiple-step-ahead forecasts, it is therefore sufficient to keep projecting ahead the particles without any weighting or resampling. In this way, the Bayesian estimation methodology provides a useful tool for calculating the full probability distribution around the median (unconditional) forecast.

4 Empirical Results

4.1 The Estimates: Static Parameters

Figure 1 plots the posterior distributions of the benchmark model’s structural parameters collected in $\theta$, obtained through the Bayesian estimation of the model’s state-space representation described in the previous section. Information about the posterior medians and the 95 percent probability intervals are also reported in Table 1, along with the summary of their prior distributions.

All in all, our results indicate that—under the assumption of fully rational expectations and serially uncorrelated disturbances—a substantial amount of habit formation and backward-lookingness has to be introduced in the prototypical small-scale DSGE model to explain the amount of persistence in actual data. In particular, our estimates indicate that the degree of habit formation in aggregate expenditure, $\eta$, and the degree of indexation to past inflation, $\gamma$, are unlikely to be lower than 0.866 and 0.837, respectively—values which are comparable with those found in the literature, despite our different estimation approach. For instance, Giannoni and Woodford (2003) estimate both $\eta = 1$ and $\gamma = 1$, suggesting an extremely high level of structural persistence in the economy. Boivin and Giannoni (2003) also find $\eta \simeq 1$ and $\gamma = 1$. Dennis (2003) estimates a new-Keynesian model with optimal monetary policy and finds $\eta \simeq 1$ and $\gamma \simeq 0.9$. Rabanal and Rubio-Ramírez (2003), by comparing different sticky-price models, estimate $\gamma \simeq 0.76$, whereas a macroeconomic study about the importance of habit formation in consumption also reports a strong role for habits, with $\eta \simeq 0.8 - 0.9$ (Fuhrer, 2000). In contrast, we find fairly weak evidence of price stickiness as measured by $\alpha$—whose posterior distribution is estimated to be centered around 0.261—a result which seems very much consistent with microeconomic data on price setting behavior (Bils and Klenow (2004)).

The output sensitivity to changes in real interest rate is quite low and loosely estimated,  

\[ \eta = \text{some value} \]  

In assessing posterior distributions, one should however keep in mind that, in order to deal with identification issues, three structural parameters have been fixed at somewhat arbitrary values. This procedure may create distortions in the estimated distribution of the remaining parameters, unless the chosen values happen to be the correct ones (Canova and Sala, 2006).
with a median intertemporal elasticity of substitution of 0.356, while estimates of the monetary policy reaction function point to a sizable degree of interest rate smoothing, with $\rho$’s 95 percent probability interval ranging between 0.791 and 0.991, two results which are quite common in previous empirical studies. Finally, the posterior distributions of $p_u$ and $p_d$ show that changes in the central bank’s target of inflation do not occur frequently, with a slight preference of downward revisions over upward revisions.

We conduct a number of robustness checks by re-estimating the model under an alternative specification of the monetary policy rule and/or making use of an alternative measure of inflation. Posterior medians and probability intervals based on a monetary policy rule reacting to deviations of output growth and/or headline inflation from equilibrium are summarized in Table 2. For convenience, medians and probability intervals for the benchmark model are also reported in the first column of the table. Although the magnitude of the changes is overall quite small, estimates appear to be sensitive to the specification adopted. In particular, the model embedding a monetary rule responding to output gap and headline inflation is found to be characterized by the lowest volatility of disturbances to trend growth and inflation. At the same time, it delivers the highest volatility of monetary disturbances, the highest degree of interest rate smoothing and price stickiness, as well as the highest degree of habit formation in consumption. At the opposite end of the spectrum, the model entailing a monetary rule responding to deviations of core inflation and output growth form equilibrium is associated with the highest volatility of shocks to trend growth and inflation, while featuring the lowest degree of interest rate smoothing, price stickiness, and habit formation.

To assess further the time series fit of our DSGE model, Figure 2 plots frequencies of draws from the posterior predictive distribution of the sample autocorrelation for the observable variables $\{y, \pi, r\}$, as well as for their deviations from corresponding equilibrium levels $\{\hat{y}, \hat{\pi}, \hat{r}\}$.

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real-time evolution of underlying trends greatly improves the fit of our DSGE model: it raises median sample autocorrelations from 0.37 to 0.99 in the case of real output, from 0.77 to 0.96 in the case of inflation, and from 0.82 to 0.90 in the case of the real interest rate. In particular, as illustrated by the middle panel of the Figure, the behavior of the natural interest rate implied by the model gives rise to an interest rate posterior predictive distribution which is remarkably consistent with the data. Yet, if one focuses on the posterior predictive distributions of real output (top panel) and inflation (bottom panel), it is quite clear that the actual autocorrelation lies outside the tail of the implied distribution, meaning that the restrictions imposed by our model forbid the exact matching of time series features. One possible explanation is that, in reality, shocks to preferences and to inflation dynamics are unlikely to be serially uncorrelated, as it is assumed in our model. Last but not least, it seems clear that for all the macroeconomic variables considered, an assumption of constant equilibrium level would be fully inconsistent with the data. Judging from the different amount of inertia characterizing the posterior distributions of "trended" and "detrended" variables, this is certainly true for real potential output. However, even in the case of inflation and real interest rates—for which the evolution of underlying trends is likely to explain a smaller fraction of the autocorrelation in actual data—the lack of overlapping between the posterior predictive distributions of "trended" and "detrended" variables point to comovements between the inflation and equilibrium inflation, on one side, and interest rate and natural interest rate, on the other one.

Summarizing, accounting for the real-time evolution of underlying trends in macroeconomic variables seems a necessary (although maybe not sufficient) condition for DSGE models to replicate important features of the data (Chang, Doh, and Schorfheide (2006), Laubach and Williams (2005a)). Absent this mechanism, there is no way that—under the assumption of rational expectations and white noise disturbances—our prototypical model could endogenously reproduce the amount of inertia inherent in the series. This is true even though a very high degree of habit formation and backward-lookingness is incorporated in the adjustment process.

4.2 The Estimates: Unobservable Time-Varying State Variables

Let us now turn the attention to the main policy issues raised at the outset. Has the Fed over 2001-03 cut interest rates by "too much" in relation to the expected return on capital? Have the latest Fed’s rate hikes completely removed policy accomodation, thereby allowing the real Fed fund rate to rise again to the natural rate of interest? Answers to these questions can be found in Figure 3, tracking down in real time the median, the 5th, and the 95th percentile of the
implied distribution of natural real rate of interest, output gap, trend growth and equilibrium inflation vis-à-vis their actual values. Our results suggest that the natural rate of interest has been steadily rising since the beginning of the 1980s, despite losing somewhat pace over latest years. At the end of the sample, the equilibrium real rate of interest is unlikely to be negative or higher than 3.2 percent, with its median value ranging between 1.3 and 1.9 percent, depending on the dataset considered. By construction, its evolution follows tightly the one of trend growth, which is estimated to hover around 3 percent, with its widest 95 percent confidence interval ranging between 1.8 and 4.1 percent.

On the one hand, our estimates of time-varying unobservable state variables show clearly how the Fed—to deal with the economic downturn following the stockmarket crash—has been cutting interest rates aggressively, thereby driving the cost of capital significantly below the natural rate for almost three years. Over the most recent quarters, however, repeated monetary policy tightening has managed to correct the disequilibrium almost completely, thus cooling down growth and bringing it back to its estimated steady-state rate.

The central bank’s success in taming inflation is also evident from the closing up of the gap between core and target inflation, since the time of the Great Moderation. It is worth noticing, however, the diverging behavior of core and headline inflation over the last few years. To examine this, we re-estimate the model using headline inflation. While there is by no means evidence of core inflationary pressures since the early 1990s (Figure 3, bottom panel), headline inflation has been crawling up since 2001 due to fast-increasing energy prices, and is estimated to be substantially (though not yet significantly) above target (Figure 4, bottom panel).

More problematic is to provide an interpretation of the 1970s Great Inflation within such a framework. It is undeniable that a generous monetary loosening was initially engineered by the Fed—by keeping interest rates significantly below the expected rate of return for quite a while—to address one of the most severe downturns ever. Certainly, one can even argue that an accomodative monetary policy stance, coupled with sizable and highly persistent supply shocks, may have induced a permanent shift in the average rate of inflation. What is difficult to explain, though, is (i) why it took so long before the drastic policy reversion could succeed in stabilizing inflation back to target, (ii) why it was so painful, and (iii) why history is not repeating itself nowadays, under very similar conditions. Unfortunately, with a rational-expectation model assuming perfect central bank’s credibility, these policy questions are deemed to remain unsolved.

Our intention is to investigate these issues in the future, within a modified framework allowing for sluggishness in expectation adjustments and nongaussianity in the specification of
shocks. For the moment, we are only interested in figuring out whether breaks in monetary policy conduct did indeed occur over the sample of interest. This is analyzed in the following subsection, by looking at the time-varying monetary policy rule parameterization of our DSGE model.

4.3 The Estimates: Time-Varying Parameters and Structural Breaks

The next step in the analysis of our benchmark DSGE model is to examine the stability of the estimated posterior parameter distributions.

In this respect, Figure 5 plots the posterior distribution of the time-varying coefficients on the Federal Reserve’s reaction function. Specifically, the upper panel graphs the median and the 95 percent confidence interval of the long-run response on core inflation, while the bottom panel reports the same percentiles for the posterior distribution of the long-run feedback on the output gap.

The estimated posterior distribution of the parameter on inflation illustrates the loss in monetary stability in the late-1970s. This response needs to be greater than one to imply monetary stability, whereas the median of its estimated distribution drops well below this threshold over the subsample 1977Q1-1980Q4, according to the model specification considered here. Strikingly, the assessment of the uncertainty around such a median estimate also shifts dramatically over the same period, with the parameter posterior distribution becoming highly asymmetric and skewed toward its lower bound. This is followed by a clear change in behavior after Paul Volker became chairman, with the median estimates of the coefficient on inflation almost doubling within two quarters and half of the distribution suddenly ranging between 1.73 and 1.82.

Movements in the response coefficient on the output gap tend to be much less evident. The corresponding median estimate is relatively stable throughout the whole sample, although characterized by a slightly negative downward trend. Nonetheless, the balance of risk associated to such a median estimate appears to have shifted starting from the second half of the seventies, with the parameter posterior distribution becoming progressively asymmetric and skewed toward its lower bound.

All in all, the estimated posterior distributions of the Fed’s monetary policy reaction function tell the conventional story of a loss of monetary control followed by a strong disinflation. Two important twists should be, however, underlined. Firstly, the evidence of shifts in monetary policy conduct becomes more or less clear-cut, depending on the assumed specification...
of the monetary policy rule itself. Figure 6 compares the posterior distributions of monetary policy responses under the four model specifications considered above. Interestingly, evidence of monetary instability is more pronounced if the monetary authority is believed to react to deviations of core inflation from equilibrium, whereas the monetary response to slacks in the economy is estimated to be less stable if the central bank focuses on an output gap measure. Indeed, the reaction function using headline inflation and output gap provides support to Orphanides' view that the Fed's inflation response in the pre- and post-Volker has not changed significantly; instead, it is the output response to have shifted under the two regimes. Secondly, in order to identify policy shifts, it is very useful to analyze changes associated with the risk assessment around (possibly similar) median responses. In this regard, the monetary policy response over the Bretton Woods period is broadly comparable with that estimated over the post-Volker period. However, the posterior distribution of the latter is remarkably more asymmetric than the former, implying greater confidence in the stability of the monetary stance.

The structural nature of the model also allows testing (i) whether the private sector's behavior has shifted over the period, and (ii) whether the typical size of idiosyncratic shocks has also changed. By comparing posterior distributions of structural parameters at the beginning of the 1980s and at the end of the sample, Table 3 and Figure 7 suggest that important shifts (even though less significant) have also characterized private sector's behavior. In particular, at the beginning of the 1980s, the predictive posterior distribution for inflation dynamics was substantially different from the one estimated at the end of the sample, featuring higher volatility of exogenous shocks ($\sigma_{\zeta^x}$) and higher backward-lookingness ($\gamma$). Conversely, the predictive posterior distribution for output dynamics was characterized by a lower degree of habit formation ($\eta$), under comparable volatility of preference shocks ($\sigma_{\zeta^\nu}$). Incidentally, the monetary policy transmission mechanism seems to have also changed. Indeed, while the median estimate of the posterior distribution associated with the intertemporal elasticity of substitution ($\sigma^{-1}$) has fallen with respect to the 1980s, gradualism in monetary policy responses ($\rho$) has increased over the same period. By placing greater reliance on the expectations channel of policy rather than on its intertemporal substitution effects, economic agents appear to have recently responded to expectations of future changes in policy rather than to those actually taking place. This interpretation would be consistent with the view that monetary policy

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18The interaction between shifts in monetary policy and changes in private sector behavior is the focus of a number of recent empirical studies (see, for instance, Bayoumi and Sgherri (2004), Benati (2004), Primiceri (2006), Lubik and Surico (2006)).
has become more effective by responding more strongly to inflation expectations (Boivin and Giannoni, 2003).

4.4 The Unconditional Forecast Distribution: Sources of Uncertainty

Figure 8 provides a visual diagnostic of the in-sample fit of our benchmark DSGE model, whose parameters’ posterior means and confidence intervals are reported in the first column of Table 2.\textsuperscript{19} The Figure plots the actual data (black lines) as well as the median of the one-period-ahead forecast distribution obtained from the filter (gray thick lines). The in-sample fit of the model is fairly good, as there seem to be no major discrepancies between actual and fitted values. However, the model has a tendency to overpredict changes in growth, inflation, and interest rates as actual variables increase and to underpredict such changes as actual variables decline. This is possibly due to the fact that the real-time estimation of the underlying stochastic processes tend to overemphasize the magnitude of slack in the economy during periods of decelerating trend growth and viceversa. The persistence of in-sample forecast errors may hence relate to inherent lags in the estimation of shifts in the underlying rate of productivity growth, given available information.

Turning to the out-of-sample predictions of the benchmark model’s state-space representation, the "fan chart" in Figure 9 illustrates the lump of uncertainty surrounding the projection of each endogenous variable at the end of the first quarter of 2006. Specifically, Figure 9 plots the full probability distribution around the central (e.g. median) forecast, which is obtained by projecting 1, 2,...,12 periods ahead the particles, on the basis of the estimated posterior distributions of parameters and unobservable state variables available at the end of the sample. Hence, in our DSGE model, the amplitude of the forecast bounds reflects not only the uncertainty associated with the parameterization of the model and the uncertainty coming from unexpected future shocks (discussed in Subsection 4.1 and 4.3), but also the uncertainty associated with the estimates of the unobservable state variables (discussed in Subsection 4.2). Overall, the resulting forecast uncertainty is substantial, but not much larger than that resulting from other DSGE models (see, for instance, Smets and Wouters, 2004), meaning that the advantage of tracking down the evolution of underlying stochastic processes did not come at the expense of worse predictive performance.

In this context, it is worth recalling that the reported forecast probability distributions define \textit{unconditional} forecast ranges. That is, implicit in our out-of-sample forecast distribu-

\textsuperscript{19}Results for the other specifications are available upon request.
tions is the assumption that the Fed will keep correcting for deviations of target variables from equilibrium, by adjusting the policy rate in line with the estimated policy rule (although with some typical errors). As a consequence, the reported out-of-sample forecasts (and corresponding forecast ranges) can be seen as representing the most likely transition path of the system back to its stochastic equilibrium, given available information at the end of the sample. Such available information relates to our updated estimates about (i) the structure of the economy (in terms of parameters and underlying trends), (ii) the volatility of the shocks, and (iii) policy preferences. Taken together, this information allows us to assess (with a certain amount of confidence) how far the economy is from equilibrium at each point in time, and to predict (with some error) the amount of disequilibrium that is likely to be corrected in each of the next $h$ periods. With this interpretation in mind, an evaluation of the forecasting performance of our DSGE model using traditional pseudo out-of-sample prediction exercises amounts to quantifying the extent to which divergences of historical data from model predictions arise from "bad initial estimates of the disequilibrium position" rather than to "bad historical shocks", which prevented the economy from moving gradually back to equilibrium.

Here, we refrain from computing root mean squared errors of pseudo out-of-sample forecasts from our DSGE model. Instead, we prefer to highlight the role of another source of forecast uncertainty, e.g. model uncertainty, to understand better the influence of alternative model specifications on the forecast.\footnote{We leave the analysis of the projections under alternative scenarios to a companion paper.} In this perspective, for each model specification considered, Table 4 reports the central projection of each of the three macroeconomic variables— as well as their projected equilibrium levels— over selected quarters of the forecast horizon 2006q2-2009q1. In line with what already discussed in previous subsections, the Table reveals that headline inflation is projected to be substantially higher than core inflation at the end of the sample (3.6 versus 2.1 percent) and is expected to fall to 3-3.2 percent by the end of the forecast horizon. The disinflation is found to be less gradual under a monetary policy reacting to headline inflation and output growth. Core inflation, on the contrary, is foreseen to remain broadly unchanged or, if anything, to increase slightly to 2.3 percent. The different projected evolution of headline and core inflation mirrors discrepancies in the estimated magnitude of disequilibrium with respect to the two measures of inflation. While headline inflation is expected to run substantially above target, core inflation is forecasted to hover at, or even below, target. In all cases, output growth is projected to be above target at the beginning of the forecast horizon and to slow down gradually over the next 12 quarters. The slowdown is more substantial under a monetary policy reacting to headline inflation and output growth. In all cases, at the beginning...
of the forecast horizon, the real interest rate is estimated to be low with respect to its natural level. The steepest rise occurs under a monetary rule reacting to headline inflation and output gap. On the contrary, under a monetary rule reacting to core inflation and output growth, we actually witness a further relaxation of the monetary policy stance over the projected horizon.

Summarizing, unconditional forecast distributions appear to be sensitive to the model specification adopted. Overall, the higher the estimated magnitude of the initial disequilibria and the higher the degree of persistence estimated in the economy, the greater the role the monetary authority is expected to play in correcting such disequilibria looking forward.

5 Conclusions and Prospects

This paper is a first attempt to provide real-time policy assessment using a dynamic optimizing business cycle model, which ensures consistency between the specification of price adjustments and the evolution of the economy under flexible prices and is generalized to allow for non-stationarity in the underlying stochastic processes. Accounting for the real-time evolution of underlying trends in productivity and real output seems capable of matching the empirical properties of macroeconomic data, thus having the potential for improving the fit and the forecasting performance of rational-expectation monetary models.

To fit our DSGE model to the data, its likelihood function is evaluated using a Sequential Monte Carlo algorithm—or particle filter—providing joint estimates of the structural parameters and the unobservable, time-varying, nonstationary state variables. To our knowledge, this is a very innovative way to bring DSGE models to data. Up to now, in the setting of DSGE models, particle filters have been used to construct the likelihood, but not to perform parameter estimation. We propose an indirect way of proceeding, which yields parameter estimates directly from the filtering procedure through a single solution of the DSGE model for each particle, thereby drastically reducing the computational burden otherwise needed to solve the model for each particle and for each time stamp.

Importantly, the paper illustrates how the posterior distribution of the model can be used to carry out predictive checks, to test for parameter instability and heteroscedasticity, as well as to construct forecast ranges around a central prediction while allowing to disentangle different sources of uncertainty.

Despite these important contributions, we have not yet exploited the full potentiality of the particle filtering methodology. Indeed, while this paper is based on a linearized version of the model, work is ongoing to incorporate nonlinearities and nongaussian error terms. This
would require the use of higher order refinement for the solution of the rational expectation model—an area where research is very active at the moment. In addition, the identification of the deep parameters would call for nonlinear mapping from the structural DSGE model into its state-space representation, another big challenge we (and the whole literature on DSGE models) need to cope with (Beyer and Farmer, 2004; Canova and Sala, 2005).

Last but not least, we aim at extending our model to account for imperfect common knowledge and learning dynamics, in order to relax the current assumption of rational expectations on the side of the agents. The intention is to deal with real-time estimates of underlying trends and sluggishness in expectation adjustments within an unified framework, thereby allowing for two distinct sources of endogenous persistence in the economy. Conveniently, the simulation-based updating scheme adopted in this paper has the advantage to be inherently very close to a “learning mechanism” of agents’ beliefs: such a feature should essentially ease our task.
References


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Table 1: Prior and posterior distribution of static parameters: benchmark model

<table>
<thead>
<tr>
<th>Description</th>
<th>Par.</th>
<th>Core inflation &amp; Gap rule</th>
<th>Prior Distribution</th>
<th>Posterior Distribution</th>
</tr>
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<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>Density</td>
<td>Domain</td>
</tr>
<tr>
<td>Degree habit formation</td>
<td>η</td>
<td>Beta</td>
<td>[0, 1]</td>
<td>.500</td>
</tr>
<tr>
<td>Interemp. elast. substit.</td>
<td>σ⁻¹</td>
<td>Gamma</td>
<td>R⁺</td>
<td>.100</td>
</tr>
<tr>
<td>Degree inflat. indexation</td>
<td>γ</td>
<td>Beta</td>
<td>[0, 1]</td>
<td>.900</td>
</tr>
<tr>
<td>Fraction non-optim. firms</td>
<td>α</td>
<td>Beta</td>
<td>[0, 1]</td>
<td>.200</td>
</tr>
<tr>
<td>Interest rate smoothing</td>
<td>ρ</td>
<td>Beta</td>
<td>[0, 1]</td>
<td>.900</td>
</tr>
<tr>
<td>Inflat. target: prob_up</td>
<td>pu</td>
<td>Beta</td>
<td>[0, 1]</td>
<td>.070</td>
</tr>
<tr>
<td>Inflat. target: prob_down</td>
<td>pd</td>
<td>Beta</td>
<td>[0, 1]</td>
<td>.120</td>
</tr>
<tr>
<td>Drift prod. shock: std. dev.</td>
<td>10⁴ · σζ*</td>
<td>Lognor.</td>
<td>R⁺</td>
<td>1.80</td>
</tr>
<tr>
<td>Prefer. shock: std. dev.</td>
<td>10² · σζ#</td>
<td>Lognor.</td>
<td>R⁺</td>
<td>.960</td>
</tr>
<tr>
<td>Supply shock: std. dev.</td>
<td>σζ⁺</td>
<td>Lognor.</td>
<td>R⁺</td>
<td>1.14</td>
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<tr>
<td>Monetary shock: std. dev.</td>
<td>σζ⁻</td>
<td>Lognor.</td>
<td>R⁺</td>
<td>1.16</td>
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</table>

Table 2: Posterior distributions of static parameters: benchmark vs. alternative specifications

<table>
<thead>
<tr>
<th>Par.</th>
<th>Core &amp; Gap rule</th>
<th>Headline &amp; Gap rule</th>
<th>Headline &amp; Grt rule</th>
<th>Core &amp; Grt rule</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Median</td>
<td>95% Interv.</td>
<td>Median</td>
<td>95% Interv.</td>
</tr>
<tr>
<td>σ⁻¹</td>
<td>.356 [0.062, 1.110]</td>
<td>.305 [.049, 1.127]</td>
<td>.282 [.046, 1.003]</td>
<td>.377 [.065, 1.281]</td>
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<tr>
<td>pu</td>
<td>.082 [.054, 1.111]</td>
<td>.067 [.050, .084]</td>
<td>.068 [.048, .086]</td>
<td>.080 [.058, .099]</td>
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<tr>
<td>10² · σζ#</td>
<td>.930 [.870, .990]</td>
<td>.900 [.830, .970]</td>
<td>.940 [.890, 1.010]</td>
<td>.890 [.830, .960]</td>
</tr>
<tr>
<td>σζ⁺</td>
<td>1.024 [.968, 1.103]</td>
<td>.994 [.934, 1.076]</td>
<td>1.029 [.983, 1.089]</td>
<td>1.065 [.925, 1.110]</td>
</tr>
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</table>
Table 3: Posterior distributions of static parameters in 1980q1

<table>
<thead>
<tr>
<th>Par.</th>
<th>Core &amp; Gap rule Median</th>
<th>95% Interv.</th>
<th>Headline &amp; Gap rule Median</th>
<th>95% Interv.</th>
<th>Headline &amp; Grt rule Median</th>
<th>95% Interv.</th>
<th>Core &amp; Grt rule Median</th>
<th>95% Interv.</th>
</tr>
</thead>
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<tr>
<td>$\eta$</td>
<td>.927 [.826, .992]</td>
<td>.930 [.847, .992]</td>
<td>.947 [.874, .995]</td>
<td>.933 [.845, .992]</td>
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<tr>
<td>$\sigma^{-1}$</td>
<td>.511 [.059, 2.039]</td>
<td>.344 [.045, 1.177]</td>
<td>.172 [.023, .609]</td>
<td>.727 [.153, 1.638]</td>
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<tr>
<td>$\gamma$</td>
<td>.962 [.904, .996]</td>
<td>.960 [.904, .995]</td>
<td>.943 [.892, .985]</td>
<td>.958 [.907, .997]</td>
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<tr>
<td>$\rho$</td>
<td>.879 [.676, .979]</td>
<td>.901 [.788, .980]</td>
<td>.673 [.527, .902]</td>
<td>.818 [.673, .964]</td>
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<tr>
<td>$p_a$</td>
<td>.072 [.053, .100]</td>
<td>.071 [.055, .087]</td>
<td>.066 [.052, .081]</td>
<td>.073 [.055, .091]</td>
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</tbody>
</table>

$10^4 \cdot \sigma_{\zeta^*}$ | 1.476 [.480, 3.795] | 1.337 [.623, 2.874] | 2.209 [.715, 4.072] | 2.807 [.152, 4.960] |
$10^2 \cdot \sigma_{\zeta^{**}}$ | .950 [.890, 1.000] | 1.000 [.880, 1.090] | .990 [.940, 1.040] | .920 [.870, 1.070] |
$\sigma_{\zeta^*}$ | 1.099 [.1028, 1.144] | 1.092 [.1000, 1.154] | 1.070 [.1036, 1.123] | 1.095 [.1063, 1.146] |
$\sigma_{\zeta^{**}}$ | 1.168 [.1140, 1.206] | 1.158 [.1125, 1.190] | 1.147 [.1127, 1.197] | 1.149 [.1117, 1.189] |

Table 4: Comparing out-of-sample forecasts

<table>
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<th>Variable</th>
<th>Model</th>
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<th>3</th>
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<td>Real Interest Rate</td>
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<td>2.12</td>
<td>2.13</td>
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<td>Head &amp; Gap rule</td>
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<tr>
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<td>Head &amp; Grt rule</td>
<td>3.55</td>
<td>3.50</td>
<td>3.47</td>
<td>3.40</td>
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<td>2.23</td>
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<td>Output Growth</td>
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<td>3.36</td>
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<td>3.42</td>
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<td>3.18</td>
<td>2.76</td>
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<tr>
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<td>Head &amp; Grt rule</td>
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<td>3.52</td>
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<td>2.91</td>
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<tr>
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<td>Core &amp; Grt rule</td>
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<td>3.15</td>
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Figure 1: Parameter estimates: prior and posterior distributions
Figure 2: Sample 1st-order autocorrelation and corresponding posterior predictive distributions
Figure 3: Estimated time-varying unobservable state variables (core inflation & gap rule)
Figure 4: Estimated time-varying unobservable state variables (headline inflation and gap rule)
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